

DUALIZABILITY OF AUTOMATIC ALGEBRAS

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ABSTRACT. We make a start on one of George McNulty’s *Dozen Easy Problems*: “Which finite automatic algebras are dualizable?” We give some necessary and some sufficient conditions for dualizability. For example, we prove that a finite automatic algebra is dualizable if its letters act as an abelian group of permutations on its states. To illustrate the potential difficulty of the general problem, we exhibit an infinite ascending chain $\mathbf{A}_1 \leq \mathbf{A}_2 \leq \mathbf{A}_3 \leq \dots$ of finite automatic algebras that are alternately dualizable and non-dualizable.

1. INTRODUCTION

In this paper, we shall make a start on Problem 6 from George McNulty’s *Dozen Easy Problems* [17]: “Which finite automatic algebras are dualizable?”

An *automatic algebra* is a set with binary operation $\mathbf{A} = \langle Q \cup \Sigma \cup \{0\}; \cdot \rangle$ that encodes a partial automaton with state set Q and alphabet Σ : the multiplication satisfies

$$q \cdot a = r \iff q \xrightarrow{a} r,$$

for all $q, r \in Q$ and $a \in \Sigma$; all other products give the default element $0 \notin Q \cup \Sigma$. The example featured in McNulty’s problem is given in Figure 1.

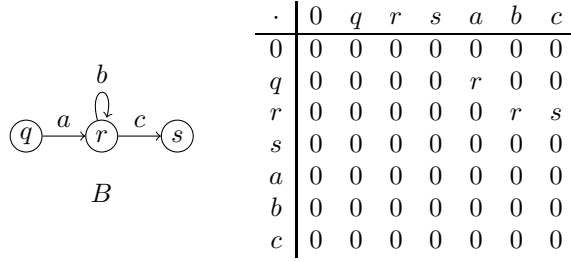


FIGURE 1. An example of an automatic algebra

Automatic algebras have been studied mostly as a source of finite algebras with non-finitely based equational theories. The first finite algebra shown to have a non-finitely based equational theory, due to Lyndon [14] in 1954, is the automatic algebra based on the automaton L pictured in Figure 2. Automatic algebras were probably first identified as a “nice class” of algebras by Kearnes and Willard [12], who proved that automatic algebras are 2-step strongly solvable. They also gave a small example of an algebra from this class whose equational theory is inherently non-finitely based and has residually large models; it is the automatic algebra based on the automaton L_3^* in Figure 2. Another automatic algebra, based on the automaton R in Figure 2, has the same property and played a supporting role in the

spectacular negative solution of R. McKenzie to Tarski's finite basis problem [16] and the Quackenbush conjecture [15].

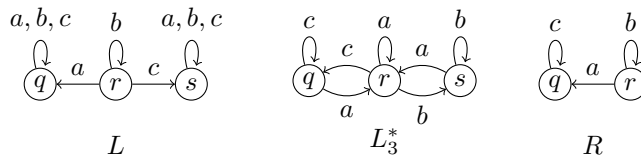


FIGURE 2. Three more examples

Automatic algebras were first named and explored systematically in the PhD theses of Z. Székely [26] and J. Boozer [1], and the article by McNulty, Székely and Willard [18]. These works provide evidence that having a finitely based equational theory is a relatively rare property amongst finite automatic algebras. Because of this, the class of finite automatic algebras may also be an interesting source of examples for studying dualizability.

A finite algebra is dualizable if it is possible (in a certain natural way) to represent the algebras in the quasi-variety $\mathbb{ISP}(\mathbf{M})$ as algebras of continuous structure-preserving maps. There is known to be a link between dualizability and residual smallness [8]: if a finite algebra is dualizable and generates a congruence-SD(\wedge) variety, then this variety is residually small. But it is unclear whether there is any link between dualizability and finite basedness. The following question, posed over 10 years ago [7], is still open: ‘Is every finite dualizable algebra finitely based?’

In this paper, we give general characterizations of dualizability within two restricted classes of finite automatic algebras: $|\Sigma| = 1$ (Theorem 6.2) and $|Q| = 2$ (Theorem 6.5). Beyond these two cases, we give several general necessary conditions for dualizability (2.5, 2.7, 2.8) and sufficient conditions for dualizability (4.1, 5.2).

All the examples of dualizable automatic algebras that we find are known to be finitely based, by Boozer [1, Theorems 1.12 and 1.16]. We shall also see that the four non-finitely based automatic algebras that encode B , L , L_3^* and R are non-dualizable; see Example 2.10. (The one based on B was shown by Boozer [1] to be non-finitely based but not inherently non-finitely based.)

The most involved proof is that of Theorem 5.2, which essentially asserts the following: if Σ acts as a coset of a subgroup of an abelian group of permutations of Q , then the automatic algebra \mathbf{M} is dualizable. We complement this theorem by giving examples of non-dualizable automatic algebras where Σ acts as a set of commuting permutations of Q (7.2, 7.3).

To illustrate the potential difficulty of McNulty’s problem, we exhibit an infinite ascending chain $\mathbf{A}_1 \leq \mathbf{A}_2 \leq \mathbf{A}_3 \leq \dots$ of finite automatic algebras that are alternately dualizable and non-dualizable (Example 7.4). This sort of bad behavior does not occur in any of the classes of finite algebras where dualizability has successfully been characterized: for example, algebras with Jónsson terms [9, 6], groups [23, 24, 19], commutative rings with unity [5], graph algebras [7] and flat graph algebras [13]. In fact, the only other such chain that has been found so far is in the class of unary algebras [20].

Notation 1.1. When working with automatic algebras, we usually indicate the groupoid operation \cdot by concatenation. Note that a groupoid term that is *not* bracketed from the left like $(\cdots((x_1x_2)x_3)x_4\cdots)x_n$ must be constantly 0 when interpreted in any automatic algebra and is therefore equivalent to the term xx . So we *always bracket from the left*. Instead of writing an expression of the form

$$(\cdots(((u \cdot v_1) \cdot v_2) \cdot v_3) \cdots) \cdot v_n,$$

we usually just write $uv_1v_2v_3\cdots v_n$, but we may choose to write $u \cdot v_1v_2v_3\cdots v_n$ or $u \cdot v_1 \cdot v_2 \cdot v_3 \cdots v_n$. We write $u \cdot v^n$ to mean $uvv\cdots v$, where the v occurs n times. Even if we use brackets, this does not override the bracket-from-the-left rule: for example, the expression $q(ab)^2$ means $qabab$, which really means $((q \cdot a) \cdot b) \cdot a \cdot b$.

We give a brief definition of ‘dualizable’ in Section 3. In the next section we do not need the definition, just the statement of the Inherent Non-dualizability Lemma. For a comprehensive introduction to natural duality theory, see [2].

2. TWO NON-DUALIZABILITY RESULTS

In this section, we give two general necessary conditions for an automatic algebra to be dualizable. We shall use the following standard technique for proving non-dualizability, due to Davey, Idziak, Lampe and McNulty [7]; see also [2, 10.5.5]. Note that a finite algebra \mathbf{M} is *inherently non-dualizable* if every finite algebra that has \mathbf{M} as a subalgebra is non-dualizable.

Lemma 2.1 (Inherent non-dualizability [7]). *Let \mathbf{M} be a finite algebra and let $\mu: \mathbb{N} \rightarrow \mathbb{N}$. Assume there is a subalgebra \mathbf{A} of \mathbf{M}^I , for some set I , and an infinite subset A_0 of A such that*

- (1) *for each $n \in \mathbb{N}$ and each congruence θ on \mathbf{A} of index at most n , the equivalence relation $\theta|_{A_0}$ has a unique block of size greater than $\mu(n)$, and*
- (2) *the algebra \mathbf{A} does not contain the element g of M^I given by $g(i) := a_i(i)$, where a_i is any element of the unique infinite block of $\ker(\pi_i)|_{A_0}$.*

Then \mathbf{M} is inherently non-dualizable.

Notation 2.2. When applying the lemma above, we use the following notation to specify elements of M^I . For all $n \in \mathbb{N}$, all distinct $i_1, \dots, i_n \in I$ and all $u, v_1, \dots, v_n \in M$, define $u_{i_1 \dots i_n}^{v_1 \dots v_n} \in M^I$ by

$$u_{i_1 \dots i_n}^{v_1 \dots v_n}(j) = \begin{cases} v_k & \text{if } j = i_k, \text{ for some } k \in \{1, \dots, n\}, \\ u & \text{otherwise.} \end{cases}$$

For $v \in M$, we define $\underline{v} \in M^I$ to be the constant map with value v .

Definition 2.3. Fix an automatic algebra $\mathbf{M} = \langle Q \cup \Sigma \cup \{0\}; \cdot \rangle$ and let $a \in \Sigma$. We shall say that the letter a acts as *whiskery cycles* if, for all $q \in Q$, there exists $n \in \mathbb{N}$ such that $qa = qa^{n+1}$. Informally, this means that each state in Q is either

- in an a -cycle,
- only one step away from an a -cycle, or
- not in the domain of a .

See Figure 3 for an example of a letter acting as whiskery cycles.

Lemma 2.4. *Let \mathbf{M} be a finite automatic algebra. The following are equivalent:*

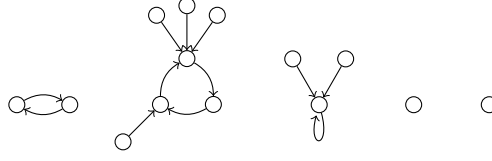


FIGURE 3. An example of whiskery cycles

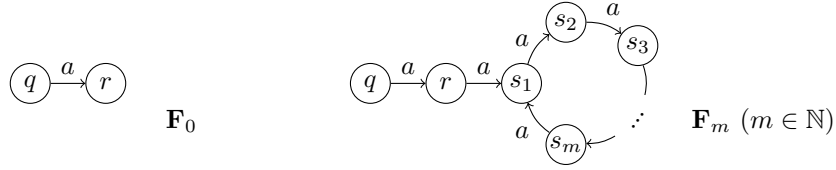


FIGURE 4. Forbidden subalgebras for whiskery cycles

- (1) *each letter acts as whiskery cycles;*
- (2) \mathbf{M} *satisfies the quasi-equation* $vxx \approx wx \implies vx \approx wx$;
- (3) *for each* $m \in \mathbb{N} \cup \{0\}$, *the automatic algebra* \mathbf{F}_m *does not embed into* \mathbf{M} ; *see Figure 4.*

Sketch proof. (1) \Rightarrow (2): Assume that each letter acts as whiskery cycles. Let $v, w, x \in M$ and assume that $vxx = wx$ in \mathbf{M} . There are $m, n \in \mathbb{N}$ such that $vx = vxx^m$ and $wx = wx^n$. So $vx = vxx^{mn} = wx^{mn} = wx$.

(2) \Rightarrow (3): The algebra \mathbf{F}_0 fails the quasi-equation, as $qaa = 0 = raa$ but $qa = r \neq 0 = ra$. Now let $m \in \mathbb{N}$. Then \mathbf{F}_m fails the quasi-equation, as there exists $k \in \{1, \dots, m\}$ such that $qaa = s_1 = s_kaa$ but $qa = r \neq s_ka$.

(3) \Rightarrow (1): Assume that a does not act as whiskery cycles. Then there is $q \in Q$ such that $qa \neq qa^{n+1}$, for all $n \in \mathbb{N}$. So $qa \neq 0$. If there is some $k \in \mathbb{N}$ such that $qa^k = 0$, then \mathbf{F}_0 embeds into \mathbf{M} . Otherwise, since \mathbf{M} is finite, there is some $m \in \mathbb{N}$ such that \mathbf{F}_m embeds into \mathbf{M} . \square

The next theorem tells us that, if a finite automatic algebra is dualizable, then every letter must act as whiskery cycles.

Theorem 2.5. *Let \mathbf{M} be a finite automatic algebra and let $a \in \Sigma$. If a does not act as whiskery cycles, then \mathbf{M} is inherently non-dualizable.*

Proof. Fix $m \in \mathbb{N} \cup \{0\}$. By Lemma 2.4, (3) \Rightarrow (1), it suffices to prove that the automatic algebra $\mathbf{F}_m = \langle Q \cup \Sigma \cup \{0\}; \cdot \rangle$ in Figure 4 is inherently non-dualizable, where $Q = \{q, r, s_1, s_2, \dots, s_m\}$ and $\Sigma = \{a\}$. (If $m = 0$, then $Q = \{q, r\}$.)

We will use Lemma 2.1 with $\mu: \mathbb{N} \rightarrow \mathbb{N}$ given by $\mu(n) := n$. Using Notation 2.2, define $A_0, B \subseteq (F_m)^\mathbb{N}$ by

$$A_0 := \{0_{1i}^{rr} \mid i \geq 2\} \quad \text{and} \quad B := \{0_{1ij}^{qqq} \mid j > i \geq 2\} \cup \{a_i^0 \mid i \geq 2\},$$

and define $A := \text{sg}_{(F_m)^\mathbb{N}}(A_0 \cup B)$. Condition 2.1(2) holds, as $g = 0_1^r$ and

$$A \subseteq A_0 \cup B \cup \{0_{1ij}^{rrr} \mid j > i \geq 2\} \cup \{0, s_1, s_2, \dots, s_m\}^\mathbb{N}.$$

It remains to establish condition 2.1(1).

Let $n \in \mathbb{N}$ and let θ be a congruence on \mathbf{A} of index at most n . We want to show that $\theta|_{A_0}$ has a unique block of size greater than n . So consider disjoint subsets J and K of $\mathbb{N} \setminus \{1\}$ with $|J| = |K| = n + 1$. Suppose that each of the two subsets $\{0_{1j}^{rr} \mid j \in J\}$ and $\{0_{1k}^{rr} \mid k \in K\}$ of A_0 is contained in a block of θ . It now suffices to prove that $\{0_{1i}^{rr} \mid i \in J \cup K\}$ is contained in a block of θ .

The subsets $\{a_j^0 \mid j \in J\}$ and $\{a_k^0 \mid k \in K\}$ of B each have size $n + 1$. Since θ is of index at most n , there must be distinct $i, j \in J$ and distinct $k, \ell \in K$ such that $a_i^0 \equiv_\theta a_j^0$ and $a_k^0 \equiv_\theta a_\ell^0$. We now calculate

$$0_{1j}^{rr} = 0_{1jk}^{qqq} \cdot a_k^0 \equiv_\theta 0_{1jk}^{qqq} \cdot a_\ell^0 = 0_{1jk}^{rrr}$$

in \mathbf{A} . By symmetry, we also have $0_{1k}^{rr} \equiv_\theta 0_{1jk}^{rrr}$. Thus $0_{1j}^{rr} \equiv_\theta 0_{1k}^{rr}$, and therefore the subset $\{0_{1i}^{rr} \mid i \in J \cup K\}$ is contained in a block of θ . We have shown that condition 2.1(1) holds. Hence \mathbf{F}_m is inherently non-dualizable. \square

Remark 2.6. The automatic algebra \mathbf{F}_0 is a 3-nilpotent semigroup, and is therefore also covered by M. Jackson's general result [11] that all finite proper 3-nilpotent semigroups are inherently non-dualizable.

While having whiskery cycles is necessary for the dualizability of an automatic algebra, we will see in Example 2.9 that it is not sufficient. However, we show in Section 6 that a finite automatic algebra with $|\Sigma| = 1$ is dualizable if and only if its single letter acts as whiskery cycles.

The next theorem provides another general necessary condition for dualizability, which will help with the classification of 2-state automatic algebras in Section 6.

Theorem 2.7. *If a finite automatic algebra \mathbf{M} fails the quasi-equation*

$$xy_1y_2 \dots y_m \approx 0 \implies xy_{\varphi(1)}y_{\varphi(2)} \dots y_{\varphi(m)} \approx 0, \quad (*)_\varphi$$

for some $m \in \mathbb{N}$ and some permutation φ of $\{1, 2, \dots, m\}$, then \mathbf{M} is inherently non-dualizable.

Proof. For each $m \in \mathbb{N}$, define the condition C_m on \mathbf{M} as follows:

- the quasi-equation $(*)_\varphi$ holds in \mathbf{M} , for all permutations φ of $\{1, 2, \dots, m\}$.

Then C_1 holds trivially. Now let $m \in \mathbb{N} \cup \{0\}$ and assume that C_{m+1} holds but C_{m+2} fails. We will prove that \mathbf{M} is inherently non-dualizable. By Theorem 2.5, we can assume that every letter of \mathbf{M} acts as whiskery cycles.

Each permutation of $\{1, 2, \dots, m+2\}$ can be obtained via composition from the transposition $(1\ 2)$ and the cycle $(1\ 2 \dots m+2)$. Since C_{m+2} fails, it must fail with $\varphi = (1\ 2)$ or $\varphi = (1\ 2 \dots m+2)$. We consider these two cases separately.

Case 1: $\varphi = (1\ 2)$. There exist $q \in Q$ and $a, b, c_1, \dots, c_m \in \Sigma$ such that

$$qabc_1 \dots c_m = 0 \quad \text{and} \quad r := qbac_1 \dots c_m \in Q.$$

We start by finding $p \in \mathbb{N}$ and a state $s \in Q$ such that

- (1) $qbb^p ac_1 \dots c_m = r$,
- (2) $saa^p ac_1 \dots c_m = r$, and
- (3) $qab^p ac_1 \dots c_m = 0$.

We are assuming that each letter of \mathbf{M} acts as whiskery cycles. So we can fix $p \in \mathbb{N}$ such that $qb = qbb^p$, and therefore (1) holds. We must have $qba \in Q$, by the definition of r . Since a acts as whiskery cycles, it follows that $qba = saa^p a$, for some $s \in Q$. So (2) holds.

Now suppose, by way of contradiction, that (3) fails. Then $qab^p \in Q$ and so we can define the states $s_0, s_1, \dots, s_p \in Q$ by

$$q \xrightarrow{a} s_0 \xrightarrow{b} s_1 \xrightarrow{b} s_2 \xrightarrow{b} \dots \xrightarrow{b} s_p.$$

We have $s_p a c_1 \dots c_m = qab^p a c_1 \dots c_m \neq 0$. Since condition C_{m+1} holds, this implies that $s_p c_1 \dots c_m a \neq 0$ and so $s_p c_1 \dots c_m \neq 0$. Therefore $s_{p-1} b c_1 \dots c_m \neq 0$, and using C_{m+1} again it follows that $s_{p-1} c_1 \dots c_m \neq 0$. Continuing to argue in this way, we will get $s_1 c_1 \dots c_m \neq 0$. But $s_1 = qab$, by definition, and so this contradicts our original assumption that $qabc_1 \dots c_m = 0$. Thus (3) holds.

We will prove that \mathbf{M} is inherently non-dualizable using Lemma 2.1 with the map $\mu: \mathbb{N} \rightarrow \mathbb{N}$ given by $\mu(n) := n^2$. Define the sets

$$A_0 := \{r_i^0 \mid i \in \mathbb{N}\} \subseteq M^{\mathbb{N}} \quad \text{and} \quad A := \{v \in M^{\mathbb{N}} \mid (\exists i) v(i) = 0\} \cup \Sigma^{\mathbb{N}}.$$

Clearly A is a subuniverse of $\mathbf{M}^{\mathbb{N}}$. Condition 2.1(2) holds, as $g = \underline{r} \notin A$.

To check condition 2.1(1), let $n \in \mathbb{N}$ and let θ be a congruence on \mathbf{A} of index at most n . Let J and K be disjoint subsets of \mathbb{N} with $|J| = |K| = n^2 + 1$, and assume that each of the subsets $\{r_j^0 \mid j \in J\}$ and $\{r_k^0 \mid k \in K\}$ of A_0 is contained in a block of θ . We want to prove that $\{r_i^0 \mid i \in J \cup K\}$ is contained in a block of θ .

We consider four subsets of A , each of size $n^2 + 1$:

$$\{b_j^0 \mid j \in J\}, \quad \{b_j^a \mid j \in J\}, \quad \{b_k^0 \mid k \in K\}, \quad \{b_k^a \mid k \in K\}.$$

(Note that the way a and b were originally chosen ensures they are distinct.) As θ is of index at most n , there are distinct $i, j \in J$ and distinct $k, \ell \in K$ such that the following relations hold:

$$b_i^0 \equiv_{\theta} b_j^0, \quad b_i^a \equiv_{\theta} b_j^a, \quad b_k^0 \equiv_{\theta} b_{\ell}^0, \quad b_k^a \equiv_{\theta} b_{\ell}^a.$$

Define $t := sba^p a c_1 \dots c_m \in Q \cup \{0\}$. Using equations (1)–(3), we calculate

$$\begin{aligned} r_i^0 &= q_{ik}^{0s} \cdot b_k^a \cdot (b_k^a)^p \cdot \underline{a} \cdot \underline{c}_1 \cdot \dots \cdot \underline{c}_m \\ &\equiv_{\theta} q_{ik}^{0s} \cdot b_{\ell}^a \cdot (b_k^a)^p \cdot \underline{a} \cdot \underline{c}_1 \cdot \dots \cdot \underline{c}_m = r_{ik\ell}^{0t0} \\ &= q_{ik\ell}^{0s0} \cdot b_{\ell}^0 \cdot (b_k^a)^p \cdot \underline{a} \cdot \underline{c}_1 \cdot \dots \cdot \underline{c}_m \\ &\equiv_{\theta} q_{ik\ell}^{0s0} \cdot b_k^0 \cdot (b_k^a)^p \cdot \underline{a} \cdot \underline{c}_1 \cdot \dots \cdot \underline{c}_m = r_{ik\ell}^{000} \\ &= q_{ik\ell}^{000} \cdot b_i^0 \cdot (\underline{b})^p \cdot \underline{a} \cdot \underline{c}_1 \cdot \dots \cdot \underline{c}_m \\ &\equiv_{\theta} q_{ik\ell}^{000} \cdot b_j^0 \cdot (\underline{b})^p \cdot \underline{a} \cdot \underline{c}_1 \cdot \dots \cdot \underline{c}_m = r_{ijk\ell}^{0000} \end{aligned}$$

in \mathbf{A} . Using symmetry, we obtain $r_i^0 \equiv_{\theta} r_{ijk\ell}^{0000} \equiv_{\theta} r_k^0$. So condition 2.1(1) holds, whence \mathbf{M} is inherently non-dualizable.

Case 2: $\varphi = (1 \ 2 \ \dots \ m+2)$. There are $q \in Q$ and $a, c_0, c_1, \dots, c_m \in \Sigma$ such that $qac_0 c_1 \dots c_m = 0$ and $qc_0 c_1 \dots c_m a \neq 0$. So $qc_0 a c_1 \dots c_m \neq 0$, as C_{m+1} holds. Thus \mathbf{M} also fails C_{m+2} via the transposition $(1 \ 2)$. So Case 1 applies, whence \mathbf{M} is inherently non-dualizable. \square

We can convert the syntactic condition of the previous result into more concrete conditions. For an automatic algebra \mathbf{M} and for $a \in \Sigma$, define the *domain* of a by $\text{dom}(a) := \{q \in Q \mid qa \neq 0\}$, define the *range* of a by $\text{ran}(a) := \{qa \mid q \in \text{dom}(a)\}$ and define the set of *kill states* for a by $\text{kill}(a) := Q \setminus \text{dom}(a)$.

In the following result, we use the standard notation Σ^* for the set of all words $a_1 a_2 \dots a_n$ in the alphabet Σ , where $n \in \mathbb{N} \cup \{0\}$.

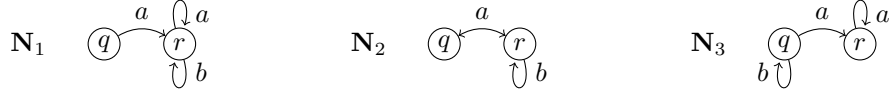


FIGURE 5. Some non-dualizable 2-state automatic algebras

Corollary 2.8. *A finite automatic algebra \mathbf{M} is inherently non-dualizable if there exists $a \in \Sigma$ such that one of the following conditions holds:*

- (1) *there is a path from the kill states of a to the domain of a , that is, there are $q \in \text{kill}(a)$ and $w \in \Sigma^*$ such that $qw \in \text{dom}(a)$;*
- (2) *there is a path from the range of a to the kill states of a , that is, there are $q \in \text{ran}(a)$ and $w \in \Sigma^*$ such that $qw \in \text{kill}(a)$.*

Proof. (1): Assume that $q \in \text{kill}(a)$ and $w \in \Sigma^*$ with $qw \in \text{dom}(a)$. Then $qaw = 0$ but $qwa \neq 0$. So \mathbf{M} is inherently non-dualizable by Theorem 2.7.

(2): Assume $q \in \text{ran}(a)$ and $w \in \Sigma^*$ with $qw \in \text{kill}(a)$. Then $qw \neq 0$ and $qwa = 0$. By Theorem 2.5, we can assume a acts as whiskery cycles. As $q \in \text{ran}(a)$, this implies that $q = qa^n$, for some $n \in \mathbb{N}$. So $qa^n w = qw \neq 0$. But $qwa = 0$ and therefore $qwa^n = 0$. Thus \mathbf{M} is inherently non-dualizable, by Theorem 2.7. \square

Example 2.9. Using the previous corollary, it is easy to check that the three automatic algebras in Figure 5 are inherently non-dualizable: both \mathbf{N}_1 and \mathbf{N}_2 have $q \in \text{kill}(b)$ but $qa \in \text{dom}(b)$, and so fail condition 2.8(1); the algebra \mathbf{N}_3 has $q \in \text{ran}(b)$ but $qa \in \text{kill}(b)$, and so fails condition 2.8(2). We use these examples in our classification of 2-state automatic algebras in Section 6.

Example 2.10. We have now covered three of the four automatic algebras from the introduction: the ones based on B and R are non-dualizable by Theorem 2.5; the one based on L_3^* is non-dualizable by Corollary 2.8, as there is a path from $s \in \text{ran}(b)$ to $q \in \text{kill}(b)$. For completeness, we shall check that the automatic algebra based on L is also non-dualizable.

Consider the subalgebra $\mathbf{M} = \langle \{q, r, s\} \cup \{a, c\} \cup \{0\}; \cdot \rangle$ of Lyndon's automatic algebra. We will use Lemma 2.1 with $\mu(n) := n$. Define $A_0 \subseteq A \subseteq M^{\mathbb{N}}$ by

$$A_0 := \{q_i^s \mid i \in \mathbb{N}\} \quad \text{and} \quad A := (Q^{\mathbb{N}} \setminus \{q, r\}^{\mathbb{N}}) \cup \Sigma^{\mathbb{N}} \cup \{0\}.$$

Then A forms a subalgebra \mathbf{A} of $\mathbf{M}^{\mathbb{N}}$, and condition 2.1(2) holds as $g = \underline{q} \notin A$.

To see that condition 2.1(1) holds, let $n \in \mathbb{N}$ and let θ be a congruence on \mathbf{A} of index at most n . Let J and K be disjoint subsets of \mathbb{N} of size $n+1$, and assume that the sets $\{q_j^s \mid j \in J\}$ and $\{q_k^s \mid k \in K\}$ are each contained in a block of θ . As θ is of index at most n , there are distinct $i, j \in J$ and distinct $k, \ell \in K$ such that $c_i^a \equiv_{\theta} c_j^a$ and $c_k^a \equiv_{\theta} c_{\ell}^a$. Therefore

$$q_i^s = q_{ik}^{sr} \cdot c_k^a \equiv_{\theta} q_{ik}^{sr} \cdot c_{\ell}^a = q_{ik}^{ss}.$$

By symmetry, we get $q_i^s \equiv_{\theta} q_{ik}^{ss} \equiv_{\theta} q_k^s$. So \mathbf{M} is inherently non-dualizable.

3. DUALIZABILITY TOOLKIT

In this section, we give some general definitions and results that will be helpful in our dualizability proofs in the following two sections. We do not need to define *dualizable* in full generality. Instead we define a simpler sufficient condition.

Definition 3.1. Fix a finite algebra \mathbf{M} . Consider a function $f: \text{hom}(\mathbf{A}, \mathbf{M}) \rightarrow M$, where \mathbf{A} is any algebra of the same type as \mathbf{M} .

- The function f is called an *evaluation* if there exists $a \in A$ with $f(x) = x(a)$, for all $x: \mathbf{A} \rightarrow \mathbf{M}$.
- For $k \in \mathbb{N}$, the function f is *k-locally an evaluation* if its restriction $f|_X$ agrees with an evaluation, for all $X \subseteq \text{hom}(\mathbf{A}, \mathbf{M})$ with $|X| \leq k$.

Now, for $k \in \mathbb{N}$, we say that \mathbf{M} is *k-dualizable* provided the following holds:

- for each finite algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ and each function $f: \text{hom}(\mathbf{A}, \mathbf{M}) \rightarrow M$, if f is *k-locally an evaluation*, then f is an evaluation.

In fact, this definition uses the Duality Compactness Theorem [28, 27, 5]; see also [2, 2.2.11]. In this paper, we always establish that a finite automatic algebra is dualizable by showing that it is *k-dualizable*, for some $k \in \mathbb{N}$. But there are dualizable algebras that are not *k-dualizable*, for any $k \in \mathbb{N}$ [21].

Definition 3.2. Let \mathbf{M} be an algebra and let $k \in \mathbb{N}$. A *k-ary relation* r on M is *compatible* with \mathbf{M} if it is a subuniverse of \mathbf{M}^k . A partial operation on M is *compatible* with \mathbf{M} if its graph is a compatible relation on \mathbf{M} (or, equivalently, its domain is a compatible relation and it is a homomorphism).

Note that relations on M can be interpreted pointwise on the subset $\text{hom}(\mathbf{A}, \mathbf{M})$ of M^A , and $\text{hom}(\mathbf{A}, \mathbf{M})$ is closed under every compatible partial operation on \mathbf{M} . We require the following easy but useful lemma (see [2, 10.5.1] or [22, 1.4.4]).

Lemma 3.3 (Preservation). *Let $k \in \mathbb{N}$ and let $f: \text{hom}(\mathbf{A}, \mathbf{M}) \rightarrow M$, where \mathbf{M} is a finite algebra and $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$. Then f is *k-locally an evaluation* if and only if f preserves every *k-ary compatible relation* on \mathbf{M} .*

We also use the fact that two different automatic algebras that generate the same quasi-variety are either both dualizable or both not.

Theorem 3.4 (Independence of the generator [10, 25]). *Let \mathbf{M} and \mathbf{N} be finite algebras and assume that $\mathbb{ISP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{N})$. If \mathbf{M} is dualizable, then so is \mathbf{N} .*

Remark 3.5. We can quickly eliminate some ‘trivial’ cases from our study of automatic algebras. If $Q = \emptyset$ or $\Sigma = \emptyset$, then the automatic algebra \mathbf{M} is a zero-semigroup and therefore dualizable (see [2, Exercise 3.7]). Also, since different automatic algebras that generate the same quasi-variety are equivalent as far as dualizability is concerned, we can make the following restrictions on the automatic algebras we consider.

- (1) *No ‘totally undefined’ letters.* Assume $Q \neq \emptyset$ and there is $a \in \Sigma$ with $\text{dom}(a) = \emptyset$. Then \mathbf{M} generates the same quasi-variety as its subalgebra \mathbf{N} with universe $N := M \setminus \{a\}$. (To see this, choose $q \in Q$ and define the embedding $\varphi: \mathbf{M} \rightarrow \mathbf{N}^2$ by $x \mapsto (x, 0)$, for all $x \in N$, and $a \mapsto (0, q)$.)
- (2) *No ‘repeated’ letters.* Assume there are distinct $a, b \in \Sigma$ such that $qa = qb$, for all $q \in Q$. Then \mathbf{M} generates the same quasi-variety as its subalgebra on $N := M \setminus \{a\}$. (Define $\varphi: M \rightarrow N^2$ by $x \mapsto (x, 0)$, for all $x \in N$, and $a \mapsto (b, b)$.)
- (3) *No ‘isolated’ states.* Assume $\Sigma \neq \emptyset$ and $q \in Q$ with $q \notin \text{dom}(a) \cup \text{ran}(a)$, for all $a \in \Sigma$. Then \mathbf{M} generates the same quasi-variety as its subalgebra on $N := M \setminus \{q\}$. (Choose $a \in \Sigma$ and define $\varphi: M \rightarrow N^2$ by $x \mapsto (x, 0)$, for all $x \in N$, and $q \mapsto (0, a)$.)

- (4) *No ‘redundant’ states.* Assume there are distinct $q, r \in Q$ with $q \notin \text{ran}(a)$ and $qa = ra$, for all $a \in \Sigma$. Then \mathbf{M} generates the same quasi-variety as its subalgebra on $N := M \setminus \{q\}$. (Define $\varphi: M \rightarrow N^2$ by $x \mapsto (x, 0)$, for all $x \in N$, and $q \mapsto (r, r)$.)

Assume \mathbf{M} is a finite automatic algebra with $M = Q \cup \Sigma \cup \{0\}$. We say that a subset C of Q is a *component* of \mathbf{M} if it is a connected component of the underlying graph of the partial automaton (that is, the graph $\langle Q; \sim \rangle$ with $q \sim r$ if and only if $qa = r$ or $ra = q$, for some $a \in \Sigma$). In this case, we call the subalgebra of \mathbf{M} with universe $C \cup \Sigma \cup \{0\}$ a *component subalgebra* of \mathbf{M} . If \mathbf{M} has only one component, then we say that it is *connected*.

The following easy fact will be useful in combination with independence of the generator (Theorem 3.4).

Lemma 3.6. *Let \mathbf{M} and \mathbf{N} be finite automatic algebras. Assume every component subalgebra of \mathbf{M} belongs to $\mathbb{ISP}(\mathbf{N})$, and vice versa. Then $\mathbb{ISP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{N})$.*

Proof. Let $\mathbf{M}_1, \dots, \mathbf{M}_n$ be the component subalgebras of \mathbf{M} . Using symmetry, it suffices to show that $\mathbf{M} \in \mathbb{ISP}(\{\mathbf{M}_1, \dots, \mathbf{M}_n\})$. For each $i \in \{1, \dots, n\}$, let C_i denote the component of \mathbf{M} corresponding to \mathbf{M}_i . So $Q = C_1 \cup \dots \cup C_n$. Now define the map $\varphi: M \rightarrow M_1 \times \dots \times M_n$ by

$$\varphi(v) = \begin{cases} (0, \dots, 0, \overset{i}{v}, 0, \dots, 0) & \text{if } v \in C_i, \text{ for some } i \in \{1, \dots, n\}, \\ (v, v, \dots, v) & \text{if } v \in \Sigma \cup \{0\}. \end{cases}$$

Then φ is an embedding from \mathbf{M} into $\mathbf{M}_1 \times \dots \times \mathbf{M}_n$. □

We now define some compatible operations and relations on automatic algebras that will be used in the following two sections.

Definition 3.7. Let \mathbf{M} be any automatic algebra. For all $u, v \in M$ such that $\{u, v\} \cap \Sigma \neq \emptyset$, we can define the homomorphism $g_{u,v}: \mathbf{M}^2 \rightarrow \mathbf{M}$ by

$$g_{u,v}(x, y) := \begin{cases} u & \text{if } (x, y) = (u, v), \\ 0 & \text{otherwise.} \end{cases}$$

To check $g_{u,v}$ is a homomorphism, let $w, x, y, z \in M$. Then $g_{u,v}(w \cdot x, y \cdot z) = 0$, as $w \cdot x, y \cdot z \in Q \cup \{0\}$, and $g_{u,v}(w, y) \cdot g_{u,v}(x, z) = 0$, as $\{0, u\} \cdot \{0, u\} = \{0\}$.

The following general lemma is an application of the ‘binary homomorphism’ techniques introduced in [3]; see also [22, Section 2.2].

Lemma 3.8. *Let \mathbf{M} be a finite algebra and let $f: \text{hom}(\mathbf{A}, \mathbf{M}) \rightarrow M$, for some finite $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$. Assume there exists $u \in \text{ran}(f)$ such that, for all $v \in M$, there is a homomorphism $g_{u,v}: \mathbf{M}^2 \rightarrow \mathbf{M}$ satisfying*

$$(\forall x, y \in M) \quad g_{u,v}(x, y) = u \iff (x, y) = (u, v).$$

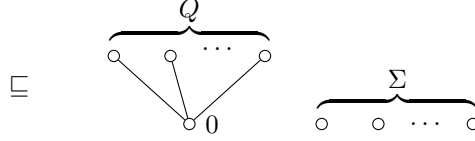
If f is 3-locally an evaluation, then f is an evaluation.

Proof. Assume f is 3-locally an evaluation. By Lemma 3.3, the map f preserves all ternary compatible relations on \mathbf{M} and therefore preserves $g_{u,u}$. We have $g_{u,u}^{-1}(u) = \{(u, u)\}$ and so, by the Strong Idempotents Lemma [3, Lemma 12], the map f agrees with evaluation at some $a \in A$ on $f^{-1}(u)$. Now let $v \in M$. Using $g_{u,v}$ and the First GST Lemma [3, Lemma 17], it follows that f also agrees with evaluation at a on $f^{-1}(v)$. Thus f is evaluation at a . □

The previous lemma and Definition 3.7 yield the following corollary, which will be used to cover one case in both of our main dualizability proofs.

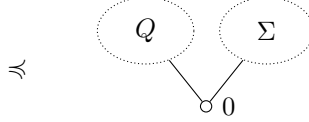
Corollary 3.9. *Let \mathbf{M} be a finite automatic algebra and let $f: \text{hom}(\mathbf{A}, \mathbf{M}) \rightarrow M$, for some finite $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$, with $\text{ran}(f) \cap \Sigma \neq \emptyset$. If f is 3-locally an evaluation, then f is an evaluation.*

Definition 3.10. Again, let \mathbf{M} be any automatic algebra. We define an order on M by $\sqsubseteq := \Delta_M \cup (\{0\} \times Q)$; see the diagram below.



The induced partial join operation is a homomorphism $\sqcup: \mathbf{D} \rightarrow \mathbf{M}$, where the domain \mathbf{D} is the subalgebra of \mathbf{M}^2 with universe $D := \sqsubseteq \cup \sqsupseteq$. To check this claim, it suffices to show that $r := \text{graph}(\sqcup)$ is a subuniverse of \mathbf{M}^3 . Let $\vec{x}, \vec{y} \in r$. Since $\hat{0} \in r$, we can assume that $\vec{x} \cdot \vec{y} \neq \hat{0}$. So there must be $q \in Q$ and $a \in \Sigma$ such that $\vec{x} \in \{(0, q, q), (q, 0, q), (q, q, q)\}$ and $\vec{y} = (a, a, a)$. Therefore $\vec{x} \cdot \vec{y} \in r$, as required.

Definition 3.11. Now let \mathbf{M} be a *total automatic algebra* (that is, an automatic algebra such that $\text{dom}(a) = Q$, for every $a \in \Sigma$). Define a quasi-order on M by $\preceq := Q^2 \cup \Sigma^2 \cup (\{0\} \times M)$; see the diagram below.



Then we can define an associated quasi-meet operation by

$$u \wedge v := \begin{cases} u & \text{if } (u, v) \in Q^2 \cup \Sigma^2, \\ 0 & \text{otherwise.} \end{cases}$$

To see that $\wedge: \mathbf{M}^2 \rightarrow \mathbf{M}$ is a homomorphism, let $x, y, u, v \in M$. We want to show that $(x \cdot u) \wedge (y \cdot v) = (x \wedge y) \cdot (u \wedge v)$. We can assume that $x, y \in Q$ and $u, v \in \Sigma$, since otherwise both sides evaluate to 0. As \mathbf{M} is total, we have $x \cdot u, y \cdot v \in Q$. So both sides evaluate to $x \cdot u$.

4. LETTERS ACTING AS CONSTANTS

In this section, we show that a finite automatic algebra is dualizable if every letter $a \in \Sigma$ acts as a constant unary operation on Q . This result will be used in Section 6, where we describe which 2-state automatic algebras are dualizable.

Note that, if every letter acts as a constant, then the automatic algebra satisfies the equation $z \cdot yx \approx z \cdot xyx$, and therefore has a finitely based equational theory by Boozer [1, Theorem 1.16].

Theorem 4.1. *Let \mathbf{M} be a finite total automatic algebra such that each letter is constant on Q . Then \mathbf{M} is dualizable.*

Proof. We can assume that $Q = \{q_1, \dots, q_n\}$ and $\Sigma = \{a_1, \dots, a_n\}$, for some $n \in \mathbb{N}$, where each letter a_i is constant with value q_i . (Use (2) and (4) from Remark 3.5. In fact, we could restrict to the case $n \leq 2$.)

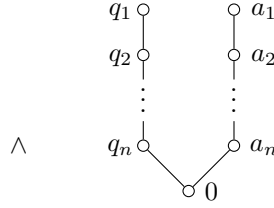
Let \mathbf{A} be a finite algebra in $\mathbb{ISP}(\mathbf{M})$ and define

$$D(\mathbf{A}) := \text{hom}(\mathbf{A}, \mathbf{M}) \subseteq M^A.$$

Assume that $f: D(\mathbf{A}) \rightarrow M$ is 4-locally an evaluation. We aim to prove that f is an evaluation.

If $\text{ran}(f) = \{0\}$, then f is given by evaluation at $0^{\mathbf{A}}$. If $\text{ran}(f) \cap \Sigma \neq \emptyset$, then f is an evaluation, by Corollary 3.9. So we can assume that $\text{ran}(f) \subseteq Q \cup \{0\}$ and, without loss of generality, that $q_1 \in \text{ran}(f)$.

We claim that the meet operation shown below is a homomorphism $\wedge: \mathbf{M}^2 \rightarrow \mathbf{M}$.



To check this claim, let $x, y, u, v \in M$. We want to show that $(x \cdot u) \wedge (y \cdot v) = (x \wedge y) \cdot (u \wedge v)$. We can assume $x, y \in Q$ and $u, v \in \Sigma$, since otherwise both sides evaluate to 0. It is now easy to check that both sides evaluate to q_m , where m is the largest index such that $a_m \in \{u, v\}$.

So $D(\mathbf{A})$ is a semilattice under the pointwise operation \wedge and the map f is a semilattice homomorphism (as f is 3-locally an evaluation). Since $D(\mathbf{A})$ is finite and $q_1 \in \text{ran}(f)$, the set $f^{-1}(q_1)$ is a principal filter of $D(\mathbf{A})$. Let $w: \mathbf{A} \rightarrow \mathbf{M}$ denote the least element of $f^{-1}(q_1)$ and define

$$A_1 := w^{-1}(q_1) \subseteq A.$$

Since f is 1-locally an evaluation, we know that $A_1 \neq \emptyset$. We will be needing the following fact about A_1 .

Claim. $f(x) = x(\sigma)$, for all $x \in f^{-1}(Q)$ and all $\sigma \in A_1$.

Let $x \in f^{-1}(Q) \subseteq D(\mathbf{A})$ and let $\sigma \in A_1 = w^{-1}(q_1)$. Say that $f(x) = q_i$. There is an automorphism φ of \mathbf{M} such that $\varphi(q_i) = q_1$. Since f is 2-locally an evaluation, it preserves φ . So $\varphi \circ x \in D(\mathbf{A})$ with $f(\varphi \circ x) = \varphi(f(x)) = q_1$. Thus $\varphi \circ x \geq w$ in the semilattice $D(\mathbf{A})$. It follows that $\varphi \circ x(\sigma) \geq w(\sigma) = q_1$ and therefore $\varphi \circ x(\sigma) = q_1$. Hence $x(\sigma) = q_i$, as required.

Now suppose, by way of contradiction, that f is not an evaluation. We consider two cases.

Case 1: $w^{-1}(a_1) = \emptyset$. Let $\vee: \{0, q_1\}^2 \rightarrow \{0, q_1\}$ denote the join operation coming from the order $0 < q_1$. Define the ternary partial operation h on M with domain $D := (M \setminus \{a_1\}) \times M^2$ by

$$h(x, y, z) := \begin{cases} y \vee z & \text{if } x = q_1 \text{ and } y, z \in \{0, q_1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to check that $\mathbf{D} \leq \mathbf{M}^3$ and that $h: \mathbf{D} \rightarrow \mathbf{M}$ is a homomorphism.

Let $\sigma \in A_1$. We are supposing that f is not given by evaluation at σ . Since $\text{ran}(f) \subseteq Q \cup \{0\}$, it follows from the claim above that there is $x_\sigma \in f^{-1}(0)$ with $0 = f(x_\sigma) \neq x_\sigma(\sigma)$. We can assume that $x_\sigma(\sigma) \in Q$. (If $x_\sigma(\sigma) = b \in \Sigma$,

then use Definition 3.7 and replace x_σ by $g_{q_1, b}(w, x_\sigma)$.) Using Definition 3.11, set $y_\sigma := w \wedge x_\sigma \in D(\mathbf{A})$. Then $f(y_\sigma) = f(w) \wedge f(x_\sigma) = 0$, with $y_\sigma(\sigma) = q_1$ and $y_\sigma(A_1) \subseteq \{0, q_1\}$.

Now enumerate A_1 as $\sigma_1, \sigma_2, \dots, \sigma_k$, where $k \in \mathbb{N}$. Since $w^{-1}(a_1) = \emptyset$ by assumption in this case, we can define $z \in D(\mathbf{A})$ by

$$z := h(w, y_{\sigma_1}, h(w, y_{\sigma_2}, h(w, y_{\sigma_3}, \dots h(w, y_{\sigma_k}, y_{\sigma_k}) \dots))).$$

We get $f(z) = 0$ and $z(A_1) = \{q_1\}$. But f agrees with an evaluation on $\{w, z\}$. So this is a contradiction.

Case 2: $w^{-1}(a_1) \neq \emptyset$. We can enumerate $A_\Sigma := w^{-1}(\Sigma) = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, where $w(\alpha_\ell) = a_1$. Now define the map $\hat{\cdot} : A_1 \rightarrow A_1$ by

$$\hat{\sigma} := \sigma \cdot \alpha_1 \alpha_2 \cdots \alpha_\ell.$$

This map is well defined because $w(\alpha_\ell) = a_1$ and so, for all $\sigma \in A_1 = w^{-1}(q_1)$, we have $w(\hat{\sigma}) = w(\sigma) \cdot w(\alpha_1)w(\alpha_2) \cdots w(\alpha_\ell) = q_1$.

Now let $\sigma \in A_1$. We are supposing that f is not given by evaluation at $\hat{\sigma} \in A_1$. Using the claim, there is $x_\sigma \in f^{-1}(0)$ such that $0 = f(x_\sigma) \neq x_\sigma(\hat{\sigma})$. Since

$$0 \neq x_\sigma(\hat{\sigma}) = x_\sigma(\sigma) \cdot x_\sigma(\alpha_1)x_\sigma(\alpha_2) \cdots x_\sigma(\alpha_\ell),$$

we have $x_\sigma(\sigma) \in Q$ and $x_\sigma(A_\Sigma) \subseteq \Sigma$. Using Definition 3.11, set $y_\sigma := w \wedge x_\sigma$. Then $f(y_\sigma) = 0$ and $y_\sigma(\sigma) = q_1$. We will use the order \sqsubseteq and partial join \sqcup from Definition 3.10. Since $x_\sigma(A_\Sigma) \subseteq \Sigma$ and $y_\sigma = w \wedge x_\sigma$, it follows that $y_\sigma \sqsubseteq w$.

Again enumerate A_1 as $\sigma_1, \sigma_2, \dots, \sigma_k$. Note that the quasi-equation

$$u_1 \sqsubseteq v \ \& \ u_2 \sqsubseteq v \implies u_1 \sqcup u_2 \sqsubseteq v$$

holds on M and therefore on $D(\mathbf{A})$. Since we have shown that w is an upper bound for $y_{\sigma_1}, y_{\sigma_2}, \dots, y_{\sigma_k}$ with respect to \sqsubseteq , it follows that we can define $z := (\cdots((y_{\sigma_1} \sqcup y_{\sigma_2}) \sqcup y_{\sigma_3}) \cdots) \sqcup y_{\sigma_k}$ in $D(\mathbf{A})$. We have $f(z) = 0$ and $z(A_1) = \{q_1\}$. But f agrees with an evaluation on $\{w, z\}$. So this is a contradiction. \square

Corollary 4.2. *Let \mathbf{M} be a finite automatic algebra such that every edge is a loop (that is, such that $qa \in \{q, 0\}$, for all $q \in Q$ and $a \in \Sigma$). Then \mathbf{M} is dualizable.*

Proof. We use independence of the generator (Theorem 3.4). Let $q \in Q$. Then $\{q\}$ is a component of \mathbf{M} . By Remark 3.5(3), we can assume q is not isolated. So there is at least one $a \in \Sigma$ with $qa = q$. By Lemma 3.6 and Remark 3.5(2), we can assume every letter in Σ fixes q . So now we can assume that every letter in Σ acts as the identity on Q . By Lemma 3.6, we can assume \mathbf{M} has only one state. Thus \mathbf{M} is dualizable by Theorem 4.1. \square

5. LETTERS ACTING AS COMMUTING PERMUTATIONS

The previous section gave a dualizability result for finite total automatic algebras in which the range of each letter is as small as possible. In this section we consider the opposite extreme, that is, where each letter acts as a permutation. We are able to prove dualizability if we also assume that, on each component, the set of permutations is a coset of a subgroup of an abelian permutation group.

Note that, if the letters of an automatic algebra act as commuting permutations, then the algebra satisfies the equations $z \cdot xy \approx z \cdot yx$ and $z \cdot x^m \approx z \cdot x^n$, for some $m > n \geq 1$, and so the algebra is finitely based by Boozer [1, Theorem 1.12].

Definition 5.1. Let $\mathbf{M} = \langle Q \cup \Sigma \cup \{0\}; \cdot \rangle$ be a finite connected automatic algebra. We say that \mathbf{M} is *letter-affine* if

- (1) each $a \in \Sigma$ acts as a permutation ρ_a of Q ,
- (2) the permutations in $\{\rho_a \mid a \in \Sigma\}$ commute, and
- (3) for all $a, b, c \in \Sigma$ there exists $d \in \Sigma$ such that $\rho_a \circ \rho_b^{-1} \circ \rho_c = \rho_d$.

A finite automatic algebra is *letter-affine* if each of its component subalgebras is letter-affine.

The aim of this section is to prove the following.

Theorem 5.2. *Every letter-affine automatic algebra is dualizable.*

As special cases, we will get the following two results.

Corollary 5.3. *Let \mathbf{M} be a finite automatic algebra with $\Sigma = \{a\}$. If a acts as a permutation of Q , then \mathbf{M} is dualizable.*

Corollary 5.4. *Let \mathbf{M} be a finite automatic algebra. If Σ acts as an abelian group of permutations of Q , then \mathbf{M} is dualizable.*

Remark 5.5. We can use independence of the generator to broaden the scope of Theorem 5.2. The letters in Σ can act as partial permutations of Q provided that, on each component of \mathbf{M} , each such partial permutation is either totally defined or totally undefined. More precisely: a finite automatic algebra \mathbf{M} is dualizable if, for each component C of \mathbf{M} , the subalgebra of \mathbf{M} with universe $C \cup \Sigma_C \cup \{0\}$ is letter-affine, where $\Sigma_C := \{a \in \Sigma \mid \text{dom}(a) \cap C \neq \emptyset\}$. (Use Theorem 3.4, Lemma 3.6 and Remark 3.5 (2), (3).)

We shall say that an automatic algebra \mathbf{M} is *permutational* if every $a \in \Sigma$ acts as a permutation of Q , and that \mathbf{M} has *commuting letters* if it satisfies the equation $x \cdot yz \approx x \cdot zy$. In particular, every letter-affine automatic algebra is permutational and has commuting letters.

For the remainder of this section, we consider a fixed finite automatic algebra $\mathbf{M} = \langle Q \cup \Sigma \cup \{0\}; \cdot \rangle$ that is permutational and has commuting letters. Our aim is to prove that, if \mathbf{M} is letter-affine, then it is dualizable. Because some parts of our argument may have future use, we will not assume that \mathbf{M} is letter-affine until that assumption is needed.

Let G_1, \dots, G_n be the components of \mathbf{M} , so that $Q = G_1 \cup \dots \cup G_n$. We start by showing that each G_i can be viewed as a finite abelian group.

Claim 5.6. *For each $i \in \{1, \dots, n\}$, there is a binary operation $*$ on G_i and a map $-_{(i)}: \Sigma \rightarrow G_i$ such that*

- (1) $(G_i; *)$ is an abelian group with generating set $\Sigma_{(i)}$, and
- (2) for all $q \in G_i$ and $a \in \Sigma$, we have $q \cdot a = q * a_{(i)}$.

Proof. Since \mathbf{M} is permutational, each letter $a \in \Sigma$ acts as a permutation $\rho_{a,i}$ of G_i . Define the permutation group

$$\Pi_i := \langle \{\rho_{a,i} \mid a \in \Sigma\} \rangle \leq S_{G_i}.$$

Then Π_i is abelian, as \mathbf{M} has commuting letters. Note that, since G_i is a component of \mathbf{M} , the group Π_i induces a transitive abelian group action on G_i .

Choose a state $e_i \in G_i$ and define the map $f: \Pi_i \rightarrow G_i$ by $f(\varphi) = \varphi(e_i)$. Then f is surjective, as Π_i acts transitively on G_i . To check that f is one-to-one, let

$\varphi, \psi \in \Pi_i$ with $\varphi(e_i) = \psi(e_i)$. Then it follows easily that $\varphi = \psi$, since Π_i induces a transitive abelian group action on G_i .

Using the bijection $f: \Pi_i \rightarrow G_i$, the abelian group operation \circ on Π_i transfers to an abelian group operation $*$ on G_i . Now define the map $-_{(i)}: \Sigma \rightarrow G_i$ by

$$a_{(i)} := e_i \cdot a = \rho_{a,i}(e_i) = f(\rho_{a,i}).$$

Since $f: \Pi_i \rightarrow G_i$ is a group isomorphism and Π_i is generated by $\{\rho_{a,i} \mid a \in \Sigma\}$, it follows that G_i is generated by $\Sigma_{(i)}$. So (1) holds.

Let $q \in G_i$ and let $a \in \Sigma$. Then $q = f(\varphi) = \varphi(e_i)$, for some $\varphi \in \Pi_i$. Since the permutations in Π_i commute, we get

$$\begin{aligned} q \cdot a &= \rho_{a,i}(q) = \rho_{a,i} \circ \varphi(e_i) = \varphi \circ \rho_{a,i}(e_i) \\ &= f(\varphi \circ \rho_{a,i}) = f(\varphi) * f(\rho_{a,i}) = q * a_{(i)}. \end{aligned}$$

So (2) holds. \square

From now on, we use multiplicative notation for the groups G_1, \dots, G_n .

Definition 5.7. For each $i \in \{1, \dots, n\}$, let e_i denote the identity element of the group G_i . Define the subgroup H_i of G_i by

$$H_i := \langle \{g^{-1}h \mid g, h \in \Sigma_{(i)}\} \rangle \leq G_i.$$

Then $\Sigma_{(i)}$ is contained in a coset of H_i . So, as $\Sigma_{(i)}$ is a generating set for G_i , the group G_i/H_i is cyclic.

Claim 5.8. *The automatic algebra \mathbf{M} is letter-affine if and only if $\Sigma_{(i)}$ is a coset of H_i in G_i , for each $i \in \{1, \dots, n\}$.*

Proof. A subset S of G_i is a coset of a subgroup of G_i if and only if S is closed under the Mal'cev operation $p(x, y, z) = xy^{-1}z$. By Claim 5.6, each letter $a \in \Sigma$ acts on the group G_i as right multiplication by $a_{(i)}$. So the claim now follows easily. \square

We next introduce some helpful compatible operations on \mathbf{M} .

Definition 5.9.

- (1) For $i \in \{1, \dots, n\}$ and $g \in G_i$, the compatible unary operation λ_g on \mathbf{M} is given by

$$\lambda_g(v) = \begin{cases} gv & \text{if } v \in G_i, \\ v & \text{otherwise.} \end{cases}$$

- (2) The compatible binary partial operation \diamond on \mathbf{M} with domain $(\bigcup_{i=1}^n G_i^2) \cup \Sigma^2 \cup \{(0, 0)\}$ is given by

$$u \diamond v = \begin{cases} 0 & \text{if } u, v \in G_i \text{ with } u^{-1}v \notin H_i, \text{ for some } i \in \{1, \dots, n\}, \\ u & \text{otherwise.} \end{cases}$$

We now begin an argument which will ultimately prove that, if \mathbf{M} is letter-affine, then it is dualizable. Consider a finite algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$. Define $D(\mathbf{A}) := \text{hom}(\mathbf{A}, \mathbf{M})$ and assume that $f: D(\mathbf{A}) \rightarrow M$ is $\max(4, 2n+1)$ -locally an evaluation. We aim to prove that f is an evaluation.

If $\text{ran}(f) = \{0\}$, then f is given by evaluation at $0^{\mathbf{A}}$. Using Corollary 3.9, we can now assume that $\text{ran}(f) \subseteq Q \cup \{0\}$, with $\text{ran}(f) \cap Q \neq \emptyset$. By re-indexing the components, we can assume that $\text{ran}(f) \cap G_i \neq \emptyset$, for $i \in \{1, \dots, \ell\}$, and $\text{ran}(f) \cap G_i = \emptyset$, for $i \in \{\ell+1, \dots, n\}$.

We shall use the quasi-order \preceq on M given by Definition 3.11.

Claim 5.10. *The set $f^{-1}(Q)$ is a ‘principal filter’ of $D(\mathbf{A})$ under the quasi-order \preceq . More precisely, there exists $w \in D(\mathbf{A})$ such that, for all $x \in D(\mathbf{A})$, we have $f(x) \in Q$ if and only if $w \preceq x$. Furthermore, for each $i \in \{1, \dots, \ell\}$, there exists $w_i \in f^{-1}(e_i)$ with $w \preceq w_i \preceq w$.*

Proof. The quasi-meet operation $\wedge: \mathbf{M}^2 \rightarrow \mathbf{M}$ from Definition 3.11 is a homomorphism. So $D(\mathbf{A}) \subseteq M^A$ is closed under \wedge and the map $f: D(\mathbf{A}) \rightarrow M$ preserves \wedge (as f is 3-locally an evaluation).

Let $x, y \in f^{-1}(Q)$. Say that $f(x) = q$ and $f(y) = r$. Then

$$f(x \wedge y) = f(x) \wedge f(y) = q \wedge r = q \in Q.$$

Thus $f^{-1}(Q)$ is also closed under \wedge . Since $D(\mathbf{A})$ is finite, we can use \wedge repeatedly to obtain a ‘least’ element w of $f^{-1}(Q)$. It follows that $f(x) \in Q$ implies $w \preceq x$, for all $x \in D(\mathbf{A})$. Now assume that $x \in D(\mathbf{A})$ with $w \preceq x$. Then $w \wedge x = w$ and so $f(w) \wedge f(x) = f(w) \in Q$. This implies that $f(x) \in Q$.

Fix $i \in \{1, \dots, \ell\}$ and choose $x_i \in f^{-1}(G_i)$. Say that $f(x_i) = g \in G_i$. Then $y_i := \lambda_{g^{-1}}(x_i) \in D(\mathbf{A})$ with $f(y_i) = \lambda_{g^{-1}}(g) = e_i$, as f preserves $\lambda_{g^{-1}}$. Now define $w_i := y_i \wedge w$. Then $w_i \preceq w$ and $f(w_i) = f(y_i \wedge w) = e_i \wedge f(w) = e_i \in Q$. So $w \preceq w_i$, by the construction of w . \square

The homomorphism $w: \mathbf{A} \rightarrow \mathbf{M}$ from the claim above partitions the set A into three subsets:

$$A_Q := w^{-1}(Q), \quad A_\Sigma := w^{-1}(\Sigma), \quad \text{and} \quad A_0 := w^{-1}(0).$$

If f is an evaluation, then it must be given by evaluation at an element of A_Q , as $f(w) \in Q$. Since w is a homomorphism and \mathbf{M} is a total automatic algebra, it is easy to see that $A_Q \cdot A_\Sigma \subseteq A_Q$ in \mathbf{A} , and that all other products in \mathbf{A} belong to A_0 .

Claim 5.11. *The set A_Q is connected by A_Σ in the following sense:*

For all $\sigma, \tau \in A_Q$, we have $\sigma \cdot \alpha_1 \alpha_2 \cdots \alpha_j = \tau \cdot \beta_1 \beta_2 \cdots \beta_k$ in \mathbf{A} , for some $j, k \geq 0$ and some $\alpha_1, \alpha_2, \dots, \alpha_j, \beta_1, \beta_2, \dots, \beta_k \in A_\Sigma$.

Proof. Define $\sigma \equiv \tau$ to mean that the above relation holds. Then \equiv is an equivalence relation on A_Q , as \mathbf{M} satisfies $x \cdot yz \approx x \cdot zy$. Suppose, by way of contradiction, that \equiv is not the total relation on A_Q . Then we can partition A_Q as $B \cup C$, where $B, C \neq \emptyset$ and $B \cap C = \emptyset$, such that each of B, C is a union of \equiv -classes. It follows that $B \cdot A_\Sigma \subseteq B$ and $C \cdot A_\Sigma \subseteq C$ in \mathbf{A} .

We can now define $x, y \in D(\mathbf{A})$ by

$$x(v) = \begin{cases} w(v) & \text{if } v \notin B, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad y(v) = \begin{cases} w(v) & \text{if } v \notin C, \\ 0 & \text{otherwise.} \end{cases}$$

By Claim 5.10, we have $f(x) = 0 = f(y)$. So f does not agree with an evaluation on the subset $\{w, x, y\}$ of $D(\mathbf{A})$, which is a contradiction. \square

Claim 5.12. *Let $x \in D(\mathbf{A})$ such that $f(x) \in G_i$, for some $i \in \{1, \dots, \ell\}$. Then $x(A_Q) \subseteq G_i$.*

Proof. By Claim 5.10, we have $w \preceq x$ and therefore $x(A_Q) \subseteq Q$ and $x(A_\Sigma) \subseteq \Sigma$. Since f is 2-locally an evaluation, it agrees with an evaluation on $\{w, x\}$. So there exists $\sigma \in A_Q$ such that $x(\sigma) = f(x) \in G_i$.

Now let $\tau \in A_Q$. By Claim 5.11, we have $\sigma \cdot \alpha_1 \alpha_2 \cdots \alpha_j = \tau \cdot \beta_1 \beta_2 \cdots \beta_k$ in \mathbf{A} , for some $j, k \geq 0$ and some $\alpha_1, \alpha_2, \dots, \alpha_j, \beta_1, \beta_2, \dots, \beta_k \in A_\Sigma$. So

$$x(\sigma) \cdot x(\alpha_1)x(\alpha_2) \cdots x(\alpha_j) = x(\tau) \cdot x(\beta_1)x(\beta_2) \cdots x(\beta_k) \text{ in } \mathbf{M}.$$

Since $x(\sigma), x(\tau) \in x(A_Q) \subseteq Q$ and $x(A_\Sigma) \subseteq \Sigma$, the states $x(\sigma)$ and $x(\tau)$ must belong to the same connected component of \mathbf{M} . Hence $x(\tau) \in G_i$. \square

Claim 5.13. *If $A_\Sigma = \emptyset$, then f is an evaluation.*

Proof. Assume $A_\Sigma = \emptyset$. Then Claim 5.11 gives $|A_Q| = 1$. Say that $A_Q = \{\sigma\}$. We will check that f is given by evaluation at σ . Let $x \in D(\mathbf{A})$. Then f agrees with an evaluation on $\{w, x\}$. But this must be evaluation at σ , since we have $f(w) \in Q$ and $w^{-1}(Q) = A_Q = \{\sigma\}$. \square

By the previous claim, we can assume that $A_\Sigma \neq \emptyset$. Enumerate the set A_Σ as $\gamma_1, \gamma_2, \dots, \gamma_\kappa$. Since the groups G_1, \dots, G_n are finite, we can choose $m \in \mathbb{N}$ so that these groups all have exponent dividing m (that is, they all satisfy the equation $x^m \approx e$). Now define the map $\hat{\cdot} : A_Q \rightarrow A_Q$ by

$$\hat{\sigma} := \sigma \cdot (\gamma_1)^m \cdots (\gamma_\kappa)^m$$

and define $\hat{A}_Q := \{\hat{\sigma} \mid \sigma \in A_Q\}$.

Claim 5.14.

- (1) *We have $\hat{A}_Q \cdot A_\Sigma \subseteq \hat{A}_Q$ in \mathbf{A} .*
- (2) *Let $X \subseteq D(\mathbf{A})$ and let $\sigma \in A_Q$. If $f|_X$ agrees with evaluation at σ , then $f|_X$ also agrees with evaluation at $\hat{\sigma}$.*

Proof. Part (1) follows because $A_Q \cdot A_\Sigma \subseteq A_Q$ and \mathbf{M} satisfies $x \cdot yz \approx x \cdot zy$. For part (2), assume $f|_X$ agrees with evaluation at σ and let $x \in X$. First assume $f(x) = 0$. Then $x(\sigma) = f(x) = 0$ and it follows easily that $x(\hat{\sigma}) = 0 = f(x)$. Now assume $f(x) \neq 0$. Then $w \preceq x$. So $x(\sigma) \in Q$ and $x(A_\Sigma) \subseteq \Sigma$. Say that $x(\sigma) \in G_i$. Since the exponent of G_i divides m , it follows by Claim 5.6(2) that $x(\hat{\sigma}) = x(\sigma) = f(x)$. \square

Claim 5.15. *The set A_Σ acts transitively on \hat{A}_Q in the following sense:*

For all $\sigma, \tau \in \hat{A}_Q$, we have $\sigma \cdot \alpha_1 \alpha_2 \cdots \alpha_k = \tau$ in \mathbf{A} , for some $k \geq 0$ and some $\alpha_1, \alpha_2, \dots, \alpha_k \in A_\Sigma$.

Proof. Let $\sigma, \tau \in \hat{A}_Q$. As \mathbf{M} satisfies $x \cdot y^{2m} \approx x \cdot y^m$, it follows that $\tau \cdot \alpha^m = \tau$, for all $\alpha \in A_\Sigma$. By Claim 5.11, we have $\sigma \cdot \alpha_1 \alpha_2 \cdots \alpha_j = \tau \cdot \beta_1 \beta_2 \cdots \beta_k$ in \mathbf{A} . So $\sigma \cdot \alpha_1 \alpha_2 \cdots \alpha_j (\beta_1 \beta_2 \cdots \beta_k)^{m-1} = \tau \cdot (\beta_1 \beta_2 \cdots \beta_k)^m = \tau$ in \mathbf{A} , as required. \square

Now define the subset B of \hat{A}_Q by

$$B := \{\sigma \in \hat{A}_Q \mid (\exists x \in f^{-1}(0)) x(\sigma) \neq 0\}.$$

Note that f cannot be evaluation at any element of B . We next construct a single homomorphism $z_B \in D(\mathbf{A})$ to witness this fact.

Claim 5.16. *There exists $z_B \in D(\mathbf{A})$ such that $f(z_B) = 0$ and $z_B(B) \subseteq Q$.*

Proof. We use the order \sqsubseteq and associated partial join \sqcup from Definition 3.10. Note that $D(\mathbf{A})$ is closed under \sqcup and that f preserves \sqcup .

Fix $\sigma \in B$. We first show that there exists $z_\sigma \in f^{-1}(0)$ with $z_\sigma(\sigma) \in Q$ and $z_\sigma \sqsubseteq w$. By the definition of B , there exists $x_\sigma \in f^{-1}(0)$ with $x_\sigma(\sigma) \neq 0$. Since $\sigma \in \hat{A}_Q$, it follows easily that $x_\sigma(\sigma) \in Q$ and $x_\sigma(A_\Sigma) \subseteq \Sigma$. Thus, if we put $z_\sigma := w \wedge x_\sigma$, then $z_\sigma(\sigma) \in Q$ and $z_\sigma \sqsubseteq w$. Finally, since f preserves \wedge , we get $f(z_\sigma) = f(w \wedge x_\sigma) = f(w) \wedge f(x_\sigma) = f(w) \wedge 0 = 0$.

Now enumerate $B = \{\sigma_1, \dots, \sigma_k\}$. Because $z_{\sigma_i} \sqsubseteq w$, for all $i \in \{1, \dots, k\}$, we can define $z_B := (\dots((z_{\sigma_1} \sqcup z_{\sigma_2}) \sqcup z_{\sigma_3}) \dots) \sqcup z_{\sigma_k}$ in $D(\mathbf{A})$. Then $f(z_B) = 0$ since f preserves \sqcup , and $z_B(B) \subseteq Q$ by construction. \square

Definition 5.17. Using the homomorphisms w_i from Claim 5.10 and z_B from Claim 5.16, we define the subset C of A_Q by

$$C := \hat{A}_Q \cap \left(\bigcap_{i=1}^{\ell} w_i^{-1}(e_i) \right) \cap z_B^{-1}(0).$$

For $i \in \{1, \dots, \ell\}$, define

$$Y_i = \{y \in D(\mathbf{A}) \mid f(y) = e_i \text{ and } w \preceq y \preceq w\}.$$

Let $Y := Y_1 \cup \dots \cup Y_\ell$.

Claim 5.18. *If $f|_Y$ agrees with evaluation at some $\sigma \in C$, then f is an evaluation.*

Proof. Assume $f|_Y$ is given by evaluation at σ , for some $\sigma \in C$. Let $x \in D(\mathbf{A})$. We will check that $f(x) = x(\sigma)$.

Case 1: $f(x) = 0$. Since $\sigma \in C \subseteq z_B^{-1}(0)$, we have $\sigma \notin B$, by Claim 5.16. Since $\sigma \in C \subseteq \hat{A}_Q$, the definition of B ensures that $x(\sigma) = 0 = f(x)$.

Case 2: $f(x) \in G_i$, for some $i \in \{1, \dots, \ell\}$. Say that $f(x) = g \in G_i$. Define $y := \lambda_{g^{-1}}(x) \wedge w \preceq w$. Then $f(y) = e_i \wedge f(w) = e_i$ and so $w \preceq y$. Thus $y \in Y_i$, giving $e_i = f(y) = y(\sigma)$. By Claim 5.12, we have $x(\sigma) \in G_i$. Therefore

$$e_i = y(\sigma) = \lambda_{g^{-1}}(x(\sigma)) \wedge w(\sigma) = g^{-1}x(\sigma),$$

and so $f(x) = g = x(\sigma)$, as required. \square

Claim 5.19. *Let $X \subseteq D(\mathbf{A})$ with $|X| \leq n$. Then there exists $\sigma \in C$ such that $f|_X$ agrees with evaluation at σ .*

Proof. As f is $(2n+1)$ -locally an evaluation, there is $\tau \in A$ such that f agrees with evaluation at τ on $X' := X \cup \{w_1, \dots, w_\ell, z_B\}$. Since $w_1(\tau) = f(w_1) = e_1 \in G_1$, we have $\tau \in A_Q$. So f also agrees with evaluation at $\sigma := \hat{\tau}$ on X' , by Claim 5.14. Because $e_i = f(w_i) = w_i(\sigma)$, we get $\sigma \in w_i^{-1}(e_i)$. Because $0 = f(z_B) = z_B(\sigma)$, we get $\sigma \in z_B^{-1}(0)$. Thus $\sigma \in C$. \square

Claim 5.20. *For all $i \in \{1, \dots, \ell\}$ and $y \in Y_i$, we have $y(C) \subseteq H_i$.*

Proof. Fix $i \in \{1, \dots, \ell\}$ and $y \in Y_i$. Let $\sigma \in C$. We shall use the binary partial operation \diamond from Definition 5.9. Since $y, w_i \in Y_i$, we have $f(y), f(w_i) \in G_i$ and $w \preceq y, w_i \preceq w$. It follows from Claim 5.12 that $(y, w_i) \in \text{dom}(\diamond)$ in $D(\mathbf{A})$. So we can define $x := y \diamond w_i \in D(\mathbf{A})$ with

$$f(x) = f(y \diamond w_i) = f(y) \diamond f(w_i) = e_i \diamond e_i = e_i.$$

As $\sigma \in A_Q$, this implies that $x(\sigma) \in G_i$, using Claim 5.12 again. As $\sigma \in C$, we have $w_i(\sigma) = e_i$. Therefore

$$y(\sigma) \diamond e_i = y(\sigma) \diamond w_i(\sigma) = x(\sigma) \in G_i,$$

whence $y(\sigma) \in H_i$. \square

We remark in passing that at this point we have already accumulated enough information to prove Corollary 5.3. (Assume $|\Sigma| = 1$. For all $i \in \{1, \dots, \ell\}$, we have $|H_i| = 1$ and so $y(C) = \{e_i\}$, for all $y \in Y_i$, by Claims 5.19 and 5.20. Thus $f|_Y$ agrees with evaluation at any $\sigma \in C$, whence f is an evaluation by Claim 5.18.)

Definition 5.21.

- (1) Let \mathcal{M} denote the $Y \times C$ matrix over Q whose entry at position (y, σ) is $y(\sigma)$. For $i \in \{1, \dots, \ell\}$ and $y \in Y_i$, the row of \mathcal{M} at position y is $y|_C \in H_i^C$, by Claim 5.20. For each $\sigma \in C$, the column of \mathcal{M} at position σ belongs to $H_1^{Y_1} \times \dots \times H_\ell^{Y_\ell}$.
- (2) Partition \mathcal{M} via $Y = Y_1 \cup \dots \cup Y_\ell$, and let \mathcal{M}_i denote the corresponding $Y_i \times C$ submatrix of \mathcal{M} with entries in H_i .

Claim 5.22. *The columns of \mathcal{M} form a coset of a subgroup of $H_1^{Y_1} \times \dots \times H_\ell^{Y_\ell}$.*

Proof. It suffices to prove that the set of columns is closed under the Mal'cev operation of $H_1^{Y_1} \times \dots \times H_\ell^{Y_\ell}$. So let $\sigma, \tau, \rho \in C$ and let c_σ, c_τ, c_ρ denote the associated columns of \mathcal{M} . We want to find $\theta \in C$ such that $c_\sigma c_\tau^{-1} c_\rho = c_\theta$, computed in $H_1^{Y_1} \times \dots \times H_\ell^{Y_\ell}$.

By Claim 5.15, we can find $\alpha_1, \dots, \alpha_k \in A_\Sigma$ such that $\sigma = \tau \cdot \alpha_1 \cdots \alpha_k$ in \mathbf{A} . Define $\theta := \rho \cdot \alpha_1 \cdots \alpha_k \in A$. We will first show that $\theta \in C$. Clearly $\theta \in \widehat{A}_Q$, by Claim 5.14(1), and $z_B(\theta) = z_B(\rho) \cdot z_B(\alpha_1) \cdots z_B(\alpha_k) = 0$, as $\rho \in z_B^{-1}(0)$. So we just need to check that $w_i(\theta) = e_i$, for $i \in \{1, \dots, \ell\}$.

Let $i \in \{1, \dots, \ell\}$ and define $a_j := w_i(\alpha_j) \in \Sigma$, for each $j \in \{1, \dots, k\}$. Then

$$w_i(\theta) = w_i(\rho) \cdot a_1 \cdots a_k = e_i \cdot a_1 \cdots a_k = w_i(\tau) \cdot a_1 \cdots a_k = w_i(\sigma) = e_i.$$

Thus $\theta \in C$.

Now let $y \in Y_i$, for some $i \in \{1, \dots, \ell\}$. It remains to check that we have $y(\sigma)y(\tau)^{-1}y(\rho) = y(\theta)$ in the group G_i . Recall from Claim 5.6 that the map $-(i): \Sigma \rightarrow G_i$ satisfies $g \cdot a = ga_{(i)}$, for all $g \in G_i$ and $a \in \Sigma$. (The left side is evaluated in the automatic algebra \mathbf{M} and the right side in the group G_i .) For each $j \in \{1, \dots, k\}$, define $b_j := y(\alpha_j) \in \Sigma$ with $b_{j(i)} := (b_j)_{(i)} \in \Sigma_{(i)} \subseteq G_i$. Calculating in the abelian group G_i , we get

$$\begin{aligned} y(\sigma)y(\tau)^{-1}y(\rho) &= y(\tau \cdot \alpha_1 \cdots \alpha_k)y(\tau)^{-1}y(\rho) \\ &= y(\tau)b_{1(i)} \cdots b_{k(i)}y(\tau)^{-1}y(\rho) \\ &= y(\rho)b_{1(i)} \cdots b_{k(i)} = y(\rho \cdot \alpha_1 \cdots \alpha_k) = y(\theta), \end{aligned}$$

as required. \square

Note that the foregoing analysis assumed only that \mathbf{M} is permutational with commuting letters. At this point we introduce the further assumption that \mathbf{M} is letter-affine. So $\Sigma_{(i)}$ is a coset of H_i in G_i , by Claim 5.8. This means that $\Sigma_{(i)}$ is closed under the Mal'cev operation p_i on G_i given by $p_i(x, y, z) = xy^{-1}z$. The next general lemma shows that p_i extends to a compatible partial operation on \mathbf{M} .

Claim 5.23. *Let $i \in \{1, \dots, n\}$ and assume that $\varphi: (G_i)^k \rightarrow G_i$ is a group homomorphism with $\varphi((\Sigma_{(i)})^k) \subseteq \Sigma_{(i)}$. Then φ extends to a compatible partial operation $\psi: (G_i)^k \cup \Sigma^k \cup \{\hat{0}\} \rightarrow M$ on \mathbf{M} with $\psi(\Sigma^k) \subseteq \Sigma$.*

Proof. For $a_1, \dots, a_k \in \Sigma$, choose $\psi(a_1, \dots, a_k)$ to be any $b \in \Sigma$ such that $b_{(i)} = \varphi((a_1)_{(i)}, \dots, (a_k)_{(i)})$. \square

We shall also use this claim to extend certain group endomorphisms of H_i to compatible partial operations on \mathbf{M} . To this end, define $n_i := |G_i/H_i|$, pick a fixed element $a_i \in \Sigma_{(i)}$ and define $u_i := a_i^{n_i} \in H_i$.

Claim 5.24. *Let $\varphi \in \text{End}(H_i)$ with $\varphi(u_i) = u_i$. Then φ extends to a compatible partial operation $\psi: G_i \cup \Sigma \cup \{\hat{0}\} \rightarrow M$ on \mathbf{M} with $\psi(G_i) \subseteq G_i$ and $\psi(\Sigma) \subseteq \Sigma$.*

Proof. Using Claim 5.23, it suffices to show that φ extends to an endomorphism ξ of G_i with $\xi(\Sigma_{(i)}) \subseteq \Sigma_{(i)}$. We shall check that we can take $\xi(g) := a_i^t \varphi(h)$, where $t \in \mathbb{Z}$ and $h \in H_i$ are such that $g = a_i^t h$.

We observed in Definition 5.7 that the group G_i/H_i is cyclic. Since $\Sigma_{(i)}$ is a generating set for G_i and $a_i \in \Sigma_{(i)}$, it follows that $a_i H_i$ is a generator of G_i/H_i . So $G_i = \bigcup_{t \in \mathbb{Z}} a_i^t H_i$, where $a_i^t \in H_i$ if and only if $n_i \mid t$.

To see that ξ is well defined, let $s, t \in \mathbb{Z}$ and $h, k \in H_i$ with $a_i^s h = a_i^t k$. Then $a_i^{s-t} = kh^{-1} \in H_i$, so $n_i \mid (s-t)$. Say that $s = n_i q + t$. Then $a_i^s \varphi(h) = u_i^q a_i^t \varphi(h) = a_i^t \varphi(u_i^q h) = a_i^t \varphi(k)$. Thus ξ is well defined.

It is easy to check that ξ is an endomorphism of the group G_i . Since $\Sigma_{(i)}$ is a coset of H_i in G_i , we have $\Sigma_{(i)} = a_i H_i$. It follows that $\xi(\Sigma_{(i)}) \subseteq \Sigma_{(i)}$. \square

Claim 5.25. *For each $i \in \{1, \dots, \ell\}$,*

- (1) *the rows of \mathcal{M}_i form a subgroup of H_i^C , and*
- (2) *the rows of \mathcal{M}_i are closed under each $\varphi \in \text{End}(H_i)$ such that $\varphi(u_i) = u_i$.*

Proof. (1): Let $y, z \in Y_i$. We want to check that $(y \upharpoonright_C)(z \upharpoonright_C) \in Y_i \upharpoonright_C$, where the multiplication is computed in the group H_i^C .

As G_i is abelian, the Mal'cev operation $p: (G_i)^3 \rightarrow G_i$ is a group homomorphism. As \mathbf{M} is letter-affine, we have $p((\Sigma_{(i)})^3) \subseteq \Sigma_{(i)}$. Thus p extends to a compatible partial operation $\bar{p}: (G_i)^3 \cup \Sigma^3 \cup \{\hat{0}\} \rightarrow M$ on \mathbf{M} with $\bar{p}(\Sigma^3) \subseteq \Sigma$, by Claim 5.23.

Since $y, z \in Y_i$, it follows using Claim 5.12 that $(y, w_i, z) \in \text{dom}(\bar{p})$ in $D(\mathbf{A})$. Since f is 4-locally an evaluation, it is easy to check that $x := \bar{p}(y, w_i, z) \in Y_i$. Finally, since $w_i(C) \subseteq \{e_i\}$, we get $x(\sigma) = y(\sigma)z(\sigma)$, for all $\sigma \in C$.

(2): This part follows similarly using Claim 5.24. \square

We need the following result about finite abelian groups, whose proof is in the appendix.

Proposition 5.26. *Let H be a finite abelian group with exponent dividing m and let $u \in H$. Then there is a homomorphism $\chi: H \rightarrow \mathbb{Z}_m$ such that, for all $h \in H \setminus \{e\}$, there exists $\varphi \in \text{End}(H)$ with $\varphi(u) = u$ and $\chi(\varphi(h)) \neq 0$.*

For each $i \in \{1, \dots, \ell\}$, we can use this proposition to choose a homomorphism $\chi_i: H_i \rightarrow \mathbb{Z}_m$ such that, for all $h \in H_i \setminus \{e_i\}$, there is $\varphi \in \text{End}(H_i)$ with $\varphi(u_i) = u_i$ and $\chi_i(\varphi(h)) \neq 0$.

Definition 5.27. Define $\bar{Y} := Y_1 \times \dots \times Y_\ell$. Let $\bar{\mathcal{M}}$ denote the $\bar{Y} \times C$ matrix over \mathbb{Z}_m whose entry at position $((y_1, \dots, y_\ell), \sigma)$ is $\sum_{i=1}^\ell \chi_i(y_i(\sigma))$.

Claim 5.28.

- (1) The columns of $\overline{\mathcal{M}}$ form a coset of a subgroup of $(\mathbb{Z}_m)^{\overline{Y}}$.
- (2) The rows of $\overline{\mathcal{M}}$ form a subgroup of $(\mathbb{Z}_m)^C$.
- (3) Every row of $\overline{\mathcal{M}}$ contains at least one 0.
- (4) If $\overline{\mathcal{M}}$ has a column that is constantly 0, then f is an evaluation.

Proof. (1): Choose $\sigma, \tau, \rho \in C$. Let $\overline{c}_\sigma, \overline{c}_\tau, \overline{c}_\rho$ be the associated columns of $\overline{\mathcal{M}}$, and let c_σ, c_τ, c_ρ be the associated columns of \mathcal{M} . By Claim 5.22 there exists $\theta \in C$ such that $c_\sigma c_\tau^{-1} c_\rho = c_\theta$; using the fact that each χ_i is a group homomorphism, it is easy to show that $\overline{c}_\sigma - \overline{c}_\tau + \overline{c}_\rho = \overline{c}_\theta$, which suffices.

(2): This part follows from Claim 5.25(1).

(3): Let $(y_1, \dots, y_\ell) \in \overline{Y}$. By Claim 5.19, there is $\sigma \in C$ such that $y_i(\sigma) = e_i$, for all $i \in \{1, \dots, \ell\}$. So the row at (y_1, \dots, y_ℓ) has a 0 in the σ position.

(4): Assume that the σ -column of $\overline{\mathcal{M}}$ is constantly 0, for some $\sigma \in C$. Now let $j \in \{1, \dots, \ell\}$ and $y \in Y_j$. We shall check that $y(\sigma) = e_j$. It will then follow by Claim 5.18 that f is an evaluation.

Let $\varphi \in \text{End}(H_j)$ with $\varphi(u_j) = u_j$. It suffices to show that $\chi_j(\varphi(y(\sigma))) = 0$. By Claim 5.25(2), there is some $z \in Y_j$ such that $z|_C = \varphi(y|_C)$. Now consider $(w_1, \dots, w_{j-1}, z, w_{j+1}, \dots, w_\ell) \in \overline{Y}$. As the σ -column of $\overline{\mathcal{M}}$ is constantly 0, we get

$$0 = \chi_j(z(\sigma)) + \sum_{i \neq j} \chi_i(w_i(\sigma)) = \chi_j(z(\sigma)) + \sum_{i \neq j} \chi_i(e_i) = \chi_j(z(\sigma))$$

and so $\chi_j(\varphi(y(\sigma))) = \chi_j(z(\sigma)) = 0$, as required. \square

The proof of the next result is in the appendix.

Proposition 5.29. *Assume M is a $j \times k$ matrix over \mathbb{Z}_m whose rows form a subgroup of $(\mathbb{Z}_m)^k$, whose columns form a coset of a subgroup of $(\mathbb{Z}_m)^j$, and which is such that every row contains at least one 0. Then some column is constantly 0.*

Using this proposition and Claim 5.28, it follows that f is an evaluation. Hence we have proved that \mathbf{M} is dualizable if it is letter-affine.

6. TWO CLASSIFICATION RESULTS

In this section, we characterize dualizability within two special classes of finite automatic algebras: $|\Sigma| = 1$ and $|Q| = 2$.

Recall that the term ‘whiskery cycles’ was introduced in Definition 2.3.

Lemma 6.1. *Let \mathbf{M} be a finite automatic algebra with $\Sigma = \{a\}$. If the letter a acts as whiskery cycles, then \mathbf{M} is dualizable.*

Proof. Assume a acts as whiskery cycles. Each state of \mathbf{M} is (1) in an a -cycle, (2) only one step away from an a -cycle, or (3) not in the domain of a . Using Remark 3.5, we can assume that \mathbf{M} has no redundant or isolated states. But the states satisfying (2) are redundant, and the states satisfying (3) are isolated. Thus we can assume that a acts as a permutation of Q , and so \mathbf{M} is dualizable by Corollary 5.3. \square

Theorem 6.2 (Classification for $|\Sigma| = 1$). *Let $\mathbf{M} = \langle Q \cup \Sigma \cup \{0\}; \cdot \rangle$ be a finite automatic algebra with $|\Sigma| = 1$. Then \mathbf{M} is dualizable if and only if the letter acts as whiskery cycles (i.e., \mathbf{M} satisfies $vxx \approx wx \implies vx \approx wx$).*

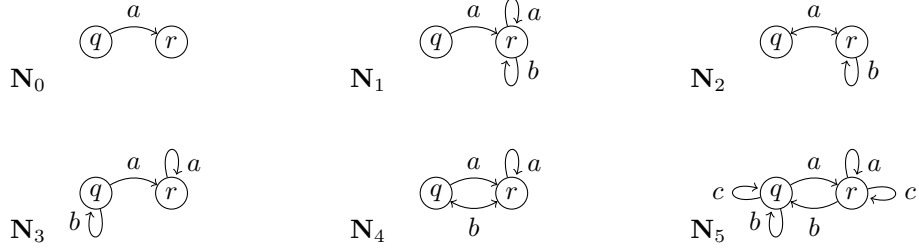


FIGURE 6. The minimal non-dualizable 2-state automatic algebras

Proof. This follows directly from Theorem 2.5 and Lemma 6.1. \square

We next complete the classification for 2-state automatic algebras. The following two algebras are not covered by any of the results we have proved so far.

Lemma 6.3. *The 2-state automatic algebra \mathbf{N}_4 from Figure 6 is inherently non-dualizable.*

Proof. We will use Lemma 2.1 with the map $\mu: \mathbb{N} \rightarrow \mathbb{N}$ given by $\mu(n) := n$. Define $A_0, A \subseteq M^{\mathbb{N}}$ by

$$A_0 := \{q_i^r \mid i \in \mathbb{N}\}, \quad A := (Q^{\mathbb{N}} \setminus \{\underline{q}\}) \cup (\Sigma^{\mathbb{N}} \setminus \{\underline{b}\}) \cup \{\underline{0}\}.$$

It is straightforward to check that A is the universe of a subalgebra \mathbf{A} of $\mathbf{M}^{\mathbb{N}}$. Condition 2.1(2) holds, as $g = \underline{q} \notin A$.

To see that condition 2.1(1) holds, let $n \in \mathbb{N}$ and let θ be a congruence on \mathbf{A} of index at most n . Consider two subsets $\{q_j^r \mid j \in J\}$ and $\{q_k^r \mid k \in K\}$ of A_0 that are each contained in a block of θ , where J and K are disjoint subsets of \mathbb{N} with $|J| = |K| = n + 1$. We want to show that $\{q_i^r \mid i \in J \cup K\}$ is contained in a block of θ .

As the sets $\{b_j^a \mid j \in J\}$ and $\{b_k^a \mid k \in K\}$ each have $n + 1$ elements, there must be distinct $i, j \in J$ and distinct $k, \ell \in K$ such that $b_i^a \equiv_{\theta} b_j^a$ and $b_k^a \equiv_{\theta} b_{\ell}^a$. We have

$$q_j^r = q_k^r \cdot b_k^a \cdot b_j^a \equiv_{\theta} q_k^r \cdot b_{\ell}^a \cdot b_j^a = q_j^{rr}.$$

A symmetric argument shows that $q_k^r \equiv_{\theta} q_{jk}^{rr}$ and hence $q_j^r \equiv_{\theta} q_k^r$. So condition 2.1(1) holds and \mathbf{M} is inherently non-dualizable. \square

Lemma 6.4. *The 2-state automatic algebra \mathbf{N}_5 from Figure 6 is inherently non-dualizable.*

Proof. We use Lemma 2.1 with $\mu: \mathbb{N} \rightarrow \mathbb{N}$ given by $\mu(n) := 1$. Let $A_0, A \subseteq M^{\mathbb{N}}$ be

$$A_0 := \{q_i^r \mid i \in \mathbb{N}\}, \quad A := (Q^{\mathbb{N}} \setminus \{\underline{q}\}) \cup (\Sigma^{\mathbb{N}} \setminus \{b, c\}^{\mathbb{N}}) \cup \{\underline{0}\}.$$

Note that A is the universe of a subalgebra \mathbf{A} of $\mathbf{M}^{\mathbb{N}}$, and that condition 2.1(2) holds, as $g = \underline{q} \notin A$.

For condition 2.1(1), let θ be a congruence on \mathbf{A} . We need to show that $\theta|_{A_0}$ has a unique non-trivial block. So assume that $q_i^r \equiv_{\theta} q_j^r$ and $q_k^r \equiv_{\theta} q_{\ell}^r$, for distinct $i, j, k, \ell \in \mathbb{N}$. It suffices to show that $q_i^r \equiv_{\theta} q_k^r$, which follows as

$$q_k^r = q_j^r \cdot b_{ik}^{ca} \equiv_{\theta} q_i^r \cdot b_{ik}^{ca} = q_{ik}^{rr}$$

and, by symmetry, $q_i^r \equiv_\theta q_{ik}^{rr}$. \square

Theorem 6.5 (Classification for $|Q| = 2$). *Let $\mathbf{M} = \langle Q \cup \Sigma \cup \{0\}; \cdot \rangle$ be a finite automatic algebra with $|Q| = 2$. Then the following are equivalent:*

- (1) \mathbf{M} is dualizable;
- (2) \mathbf{M} satisfies the equations $xy \approx xyxy$ and $wxyz \approx wyxz$;
- (3) none of the six automatic algebras in Figure 6 embeds into \mathbf{M} .

Proof. (1) \Rightarrow (3): The six algebras in Figure 6 are inherently non-dualizable by Theorem 2.5, Example 2.9, and Lemmas 6.3 and 6.4.

(2) \Rightarrow (3): The algebra \mathbf{N}_0 fails the first equation, as $qa = r \neq 0 = qaqa$. The other algebras fail the second equation, as shown below.

$$\begin{array}{ll} \mathbf{N}_1, \mathbf{N}_2: qabb = r \neq 0 = qbab; & \mathbf{N}_3: qbaa = r \neq 0 = qaba; \\ \mathbf{N}_4: qbab = q \neq r = qabb; & \mathbf{N}_5: qabc = q \neq r = qbac. \end{array}$$

(3) \Rightarrow (1) & (2): Assume $\mathbf{N}_i \not\hookrightarrow \mathbf{M}$, for all $i \in \{0, 1, \dots, 5\}$. We want to show that \mathbf{M} is dualizable and satisfies $xy \approx xyxy$ and $wxyz \approx wyxz$. By Remark 3.5, the automatic algebra \mathbf{M} generates the same quasi-variety (and therefore variety) as one with no ‘repeated letters’ and no ‘totally undefined letters’. So we can assume \mathbf{M} has no such letters, by Theorem 3.4.

Since $\mathbf{N}_0 \not\hookrightarrow \mathbf{M}$, each letter in Σ acts as

- the transposition,
- a constant, or
- a restriction of the identity.

This implies that \mathbf{M} satisfies the first equation $xy \approx xyxy$.

First assume that each edge in the partial automaton of \mathbf{M} is a loop. So \mathbf{M} is dualizable, by Corollary 4.2. Let $a, b, c \in \Sigma$ and define $D := \text{dom}(a) \cap \text{dom}(b) \cap \text{dom}(c) \subseteq Q$. Let $q \in Q$. Since every edge is a loop, if $q \in D$, then $qabc = q = qbac$, and if $q \in Q \setminus D$, then $qabc = 0 = qbac$. So \mathbf{M} satisfies the second equation.

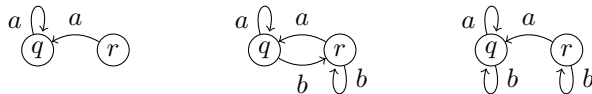
Now assume that there is an edge that is not a loop. Then Σ must contain a letter that acts as the transposition or as a constant.

First consider the case where a letter in Σ acts as the transposition on Q . Then there can be no constants (as $\mathbf{N}_4 \not\hookrightarrow \mathbf{M}$) and there can be no proper restricted identity (as $\mathbf{N}_2 \not\hookrightarrow \mathbf{M}$). So \mathbf{M} is isomorphic to one of the following two algebras.



In both cases, the algebra \mathbf{M} is dualizable (by Corollaries 5.3 and 5.4) and satisfies $wxyz \approx wyxz$.

Now we are down to the case where Σ contains a constant letter. There can be no proper restricted identity (as $\mathbf{N}_1, \mathbf{N}_3 \not\hookrightarrow \mathbf{M}$) and no transposition (as $\mathbf{N}_4 \not\hookrightarrow \mathbf{M}$). As $\mathbf{N}_5 \not\hookrightarrow \mathbf{M}$, it follows that \mathbf{M} is isomorphic to one of the following.



These three algebras all satisfy $wxyz \approx wyxz$. The first two are dualizable by Theorem 4.1. So it remains to check that \mathbf{M} is dualizable if it is the third. In this case, define \mathbf{L} to be the subalgebra of \mathbf{M} on $L := \{q, a, 0\}$. There is an embedding $\varphi: \mathbf{M} \rightarrow \mathbf{L}^2$ given by $x \mapsto (x, 0)$, for all $x \in L$, $r \mapsto (q, q)$ and $b \mapsto (a, a)$. Since \mathbf{L} is dualizable by Theorem 4.1, it follows by Theorem 3.4 that \mathbf{M} is too. \square

7. ALTERNATING CHAIN

To complement Theorem 5.2, we show that an automatic algebra \mathbf{M} can be non-dualizable if Σ acts as a set of commuting permutations of Q . We can then give an infinite ascending chain of automatic algebras that are alternately dualizable and non-dualizable.

Since we are finding non-dualizable automatic algebras that are not inherently non-dualizable, we need to use the ‘non-inherent’ version of Lemma 2.1.

Lemma 7.1 (Non-dualizability [4]). *Let \mathbf{M} be a finite algebra and let $\nu \in \mathbb{N}$. Assume there is a subalgebra \mathbf{A} of \mathbf{M}^I , for some set I , and an infinite subset A_0 of A such that*

- (1) *for each homomorphism $x: \mathbf{A} \rightarrow \mathbf{M}$, the equivalence relation $\ker(x|_{A_0})$ has a unique block of size more than ν , and*
- (2) *the algebra \mathbf{A} does not contain the element g of M^I given by $g(i) := a_i(i)$, where a_i is any element of the unique infinite block of $\ker(\pi_i|_{A_0})$.*

Then \mathbf{M} is non-dualizable.

The next theorem, which is technical in its details, can be viewed as a partial converse to Theorem 5.2. In slightly simplified terms, it states the following: let \mathbf{M} be a finite permutational automatic algebra with commuting letters, at least two of which act differently; if \mathbf{M} is dualizable, then it has a letter-affine subalgebra with at least two letters acting differently.

Theorem 7.2. *Let $\mathbf{M} = \langle Q \cup \Sigma \cup \{0\}; \cdot \rangle$ be a finite automatic algebra and let $m > 1$. Assume that*

- (a) *each $a \in \Sigma$ acts as a permutation ρ_a of Q ,*
- (b) *the permutations in $\{\rho_a \mid a \in \Sigma\}$ commute,*
- (c) *there are $b, c \in \Sigma$ such that the permutation $\rho_b(\rho_c)^{-1}$ of Q has order m ,*
- (d) *for each component C of \mathbf{M} , there is no non-trivial subgroup H of the symmetric group S_C such that $|H|$ divides m and the set $\{\rho_a|_C \mid a \in \Sigma\}$ contains a coset of H .*

Then \mathbf{M} is non-dualizable.

Proof. Define λ to be the least common multiple of the orders of the permutations ρ_b and ρ_c of Q . Throughout this proof, we blur the distinction between the elements b, c of Σ and the permutations ρ_b, ρ_c of Q : for $q \in Q$, we write $q \cdot b^{-1}$ to mean $q \cdot \rho_b^{-1}$ and write $q \cdot c^{-1}$ to mean $q \cdot \rho_c^{-1}$.

As the permutation $\rho_b(\rho_c)^{-1}$ of Q has order $m > 1$, there are distinct states $r, s \in Q$ such that

$$r = s \cdot bc^{-1}.$$

We shall use the Non-dualizability Lemma 7.1 with $\nu := |\Sigma| - 1$. (Note that $|\Sigma| \geq 2$ and so $\nu \geq 1$.) Define the index set $S := (Q \times \{b, c\}) \cup \mathbb{N}$. For each $i \in \mathbb{N}$,

define $v_i \in Q^S$ by

$$v_i(q, b) = q, \quad v_i(q, c) = q \quad \text{and} \quad v_i(j) = \begin{cases} r & \text{if } j = i, \\ s & \text{otherwise,} \end{cases}$$

for all $q \in Q$ and $j \in \mathbb{N}$. Now define $A_0 := \{v_i \mid i \in \mathbb{N}\}$. For each $I \subseteq \mathbb{N}$ with $|I| = \nu$, define $w_I \in \Sigma^S$ by

$$w_I(q, b) = b, \quad w_I(q, c) = c \quad \text{and} \quad w_I(j) = \begin{cases} b & \text{if } j \in I, \\ c & \text{otherwise,} \end{cases}$$

for all $q \in Q$ and $j \in \mathbb{N}$. Now define $B := \{w_I \mid I \subseteq \mathbb{N} \text{ with } |I| = \nu\}$ and define $A := \text{sg}_{\mathbf{M}^S}(A_0 \cup B)$.

Step 1. We first check condition 7.1(2). Note that $g \in M^S$ is given by

$$g(q, b) = q, \quad g(q, c) = q \quad \text{and} \quad g(j) = s,$$

for all $q \in Q$ and $j \in \mathbb{N}$. Suppose that $g \in A$. Then we can write

$$g = v_i \cdot w_{I_1} \cdots w_{I_\ell} \tag{1}$$

in \mathbf{M}^S . By considering equation (1) at each coordinate in $Q \times \{b, c\}$, we infer that $(\rho_b)^\ell = \text{id}_Q = (\rho_c)^\ell$. So ℓ is a multiple of λ .

Case 1: $i \notin I_1 \cup \cdots \cup I_\ell$. Since λ divides ℓ , we have $r \cdot c^\ell = r$. But evaluating equation (1) at coordinate i gives $s = r \cdot c^\ell$, which is a contradiction.

Case 2: $i \in I_1 \cup \cdots \cup I_\ell$. Enumerate $I_1 \cup \cdots \cup I_\ell$ as $i = i_0, i_1, \dots, i_k$. For each $j \in \{0, 1, \dots, k\}$, let n_j denote the number of occurrences of i_j in the sets I_1, \dots, I_ℓ . Then $n_0 + n_1 + \cdots + n_k = \nu\ell$, as the sets I_1, \dots, I_ℓ all have size ν .

Evaluating equation (1) at coordinate $i = i_0$, we get $s = r \cdot b^{n_0} c^{\ell - n_0}$, since the permutations ρ_b and ρ_c commute. This gives $s = r \cdot (bc^{-1})^{n_0}$, as λ divides ℓ . For each $j \in \{1, \dots, k\}$, by evaluating equation (1) at coordinate i_j we get $s = s \cdot b^{n_j} c^{\ell - n_j}$ and so $s = s \cdot (bc^{-1})^{n_j}$. Since λ divides ℓ , we now obtain

$$\begin{aligned} r &= r \cdot (bc^{-1})^{\nu\ell} = r \cdot (bc^{-1})^{n_0 + n_1 + \cdots + n_k} \\ &= r \cdot (bc^{-1})^{n_0} (bc^{-1})^{n_1} \cdots (bc^{-1})^{n_k} = s \cdot (bc^{-1})^{n_1} \cdots (bc^{-1})^{n_k} = s, \end{aligned}$$

which is a contradiction.

Step 2. To check condition 7.1(1), let $x: \mathbf{A} \rightarrow \mathbf{M}$ be a homomorphism. By considering three separate cases, we will show that $\ker(x|_{A_0})$ has a unique block of size more than ν . In each case, the following equation plays a central role:

$$v_i = v_j \cdot w_{K \cup \{i\}} (w_{K \cup \{j\}})^{-1}, \tag{2}$$

for all $i, j \in \mathbb{N}$ and all $K \subseteq \mathbb{N} \setminus \{i, j\}$ with $|K| = \nu - 1$.

Case 1: $0 \in x(A_0)$. By equation (2), we must have $x(A_0) = \{0\}$.

Case 2: $x(A_0) \cap \Sigma \neq \emptyset$. By equation (2), we get $x(A_0) = \{0\}$. So this case cannot happen.

Case 3: $x(A_0) \subseteq Q$. By equation (2), we have $x(B) \subseteq \Sigma$ and $x(A_0) \subseteq C$, for some component C of \mathbf{M} . Let $\gamma: \Sigma \rightarrow S_C$ map each letter to its action on C ; so that $\gamma(a) = \rho_a|_C$. Then $\langle \gamma(\Sigma) \rangle$ is a transitive abelian group of permutations of C , and is therefore regular (i.e., the stabilizer of each element of C is trivial). For all $w_I \in B \subseteq \{b, c\}^S$, we have $v_1 = v_1 \cdot (w_I)^\lambda$ in \mathbf{A} and so $x(v_1) = x(v_1) \cdot x(w_I)^\lambda$ in \mathbf{M} ;

it follows that the order of the permutation $\gamma(x(w_I))$ of C divides λ ; this means that, for $q \in C$, it makes sense to write $q \cdot x(w_I)^{-1}$ to mean $q \cdot x(w_I)^{\lambda-1}$.

Assume that $I = \{i_1, \dots, i_{\nu+1}\}$ and $J = \{j_1, \dots, j_{\nu+1}\}$ are disjoint subsets of \mathbb{N} , each of size $\nu + 1$, such that

$$x(v_{i_1}) = x(v_{i_2}) = \dots = x(v_{i_{\nu+1}}) \quad \text{and} \quad x(v_{j_1}) = x(v_{j_2}) = \dots = x(v_{j_{\nu+1}}).$$

It suffices to show that $x(v_{i_1}) = x(v_{j_1})$.

First define a sequence of $\nu + 2$ subsets of \mathbb{N} :

$$\begin{aligned} I_0 &:= \{i_1, i_2, i_3, \dots, i_{\nu+1}\} = I, \\ I_1 &:= \{j_1, i_2, i_3, \dots, i_{\nu+1}\}, \\ I_2 &:= \{j_1, j_2, i_3, \dots, i_{\nu+1}\}, \\ &\vdots \\ I_{\nu+1} &:= \{j_1, j_2, j_3, \dots, j_{\nu+1}\} = J. \end{aligned}$$

We shall use the following consequence of equation (2):

$$v_{i_1} \equiv_x v_{i_{n+1}} = v_{j_{n+1}} \cdot w_{I_n}(w_{I_{n+1}})^{-1} \equiv_x v_{j_1} \cdot w_{I_n}(w_{I_{n+1}})^{-1}, \quad (3)$$

for all $n \in \{0, 1, \dots, \nu\}$. This implies that

$$x(v_{j_1}) \cdot x(w_{I_0})x(w_{I_1})^{-1} = x(v_{j_1}) \cdot x(w_{I_n})x(w_{I_{n+1}})^{-1},$$

for all $n \in \{1, \dots, \nu\}$. (Recall that the order of each permutation of C in $\gamma(x(B))$ divides λ .) As $\langle \gamma(\Sigma) \rangle$ is a regular group of permutations of C , it follows that

$$h := \gamma(x(w_{I_1}))^{-1} \circ \gamma(x(w_{I_0})) = \gamma(x(w_{I_{n+1}}))^{-1} \circ \gamma(x(w_{I_n})), \quad (4)$$

for all $n \in \{1, \dots, \nu\}$.

As $x(B) \subseteq \Sigma$ and $|\Sigma| = \nu + 1$, we have $\gamma(x(w_{I_k})) = \gamma(x(w_{I_\ell}))$, for some $k < \ell$. We may choose k, ℓ with $\ell - k$ minimal. Now consider the distinct permutations

$$\gamma(x(w_{I_k})), \gamma(x(w_{I_{k+1}})), \dots, \gamma(x(w_{I_{\ell-1}})) \quad (5)$$

of C . By (4), we have

$$\gamma(x(w_{I_n})) \circ h^{-1} = \gamma(x(w_{I_{n+1}})),$$

for all $n \in \{k, \dots, \ell - 1\}$, and

$$\gamma(x(w_{I_{\ell-1}})) \circ h^{-1} = \gamma(x(w_{I_\ell})) = \gamma(x(w_{I_k})).$$

It follows that the permutations in (5) form a coset of the cyclic subgroup $\langle h \rangle$ of S_C . So the permutation h of C has order $\ell - k$.

As the permutation $\rho_b(\rho_c)^{-1}$ of Q has order m , we have $v_1 \cdot (w_{I_0}(w_{I_1})^{-1})^m = v_1$ in \mathbf{A} , and therefore

$$h^m(x(v_1)) = x(v_1) \cdot (x(w_{I_0})x(w_{I_1})^{-1})^m = x(v_1)$$

in \mathbf{M} . As $\langle \gamma(\Sigma) \rangle$ is regular, this implies that the order of h divides m . So $\ell - k$ divides m . By assumption, this is only possible if the subgroup $\langle h \rangle$ of S_C is trivial, which implies that $h = \text{id}_C$. Hence $\gamma(x(w_{I_0})) = \gamma(x(w_{I_1}))$, by (4). Now, using (3), we have

$$x(v_{j_1}) = x(v_{j_1}) \cdot x(w_{I_0})x(w_{I_1})^{-1} = x(v_{j_1} \cdot w_{I_0}(w_{I_1})^{-1}) = x(v_{i_1}).$$

So 7.1(1) holds, as required. \square

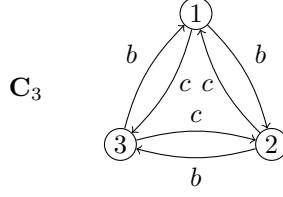


FIGURE 7. A non-dualizable automatic algebra

Example 7.3. The simplest example coming from the previous theorem is the 3-state automatic algebra $\mathbf{C}_3 = \langle \{1, 2, 3\} \cup \{b, c\} \cup \{0\}; \cdot \rangle$ shown in Figure 7. This algebra is non-dualizable by Theorem 7.2. But \mathbf{C}_3 is not inherently non-dualizable, by Corollary 5.4: a dualizable automatic algebra can be obtained from \mathbf{C}_3 by adding a letter that acts as the identity.

We can now give the promised alternating chain.

Example 7.4. *There is an infinite ascending chain $\mathbf{M}_1 \leq \mathbf{M}_2 \leq \mathbf{M}_3 \leq \dots$ of finite automatic algebras that are alternately dualizable and non-dualizable.*

Proof. For each odd prime p , let $\mathbf{C}_p = \langle \{1, 2, \dots, p\} \cup \{b, c\} \cup \{0\}; \cdot \rangle$ be the p -state version of the 3-state automatic algebra from Figure 7. We start with $\mathbf{M}_1 := \mathbf{C}_3$. So \mathbf{M}_1 is non-dualizable, by the previous example.

Now assume $\mathbf{M}_n = \langle Q_n \cup \Sigma_n \cup \{0\}; \cdot \rangle$ has been defined, for some odd number n , so that Σ_n consists of commuting permutations of Q_n . To create \mathbf{M}_{n+1} , take $Q_{n+1} := Q_n$ and construct Σ_{n+1} from Σ_n by adding enough new permutations so that Σ_{n+1} forms an abelian group of permutations of Q_{n+1} . Then \mathbf{M}_{n+1} is dualizable, by Corollary 5.4.

Finally, assume that $\mathbf{M}_n = \langle Q_n \cup \Sigma_n \cup \{0\}; \cdot \rangle$ has been defined, for some even number n , so that Σ_n consists of commuting permutations of Q_n . Choose a prime $p > |\Sigma_n| + 3$. Define $Q_{n+1} := Q_n \dot{\cup} \{1, 2, \dots, p\}$ and $\Sigma_{n+1} := \Sigma_n \dot{\cup} \{b, c\}$. In \mathbf{M}_{n+1} , each letter in Σ_n should act on Q_n as it does in \mathbf{M}_n and act on $\{1, 2, \dots, p\}$ as the identity, and each letter in $\{b, c\}$ should act on Q_n as the identity and act on $\{1, 2, \dots, p\}$ as it does in \mathbf{C}_p . So the action corresponding to bc^{-1} has order p .

We will use the previous theorem to check that \mathbf{M}_{n+1} is non-dualizable. Let D be a component of \mathbf{M}_{n+1} . If $D = \{1, 2, \dots, p\}$, then $|\{\rho_a \upharpoonright_D \mid a \in \Sigma_{n+1}\}| = 3 < p$. If $D \subseteq Q_n$, then

$$|\{\rho_a \upharpoonright_D \mid a \in \Sigma_{n+1}\}| \leq |\{\rho_a \upharpoonright_D \mid a \in \Sigma_n\}| + 1 < p,$$

since $|\Sigma_n| + 1 < p$. It follows that \mathbf{M}_{n+1} is non-dualizable. \square

8. APPENDIX

This appendix contains proofs of the two purely group-theoretic results used in Section 5.

Proposition. *Let H be a finite abelian group with exponent dividing m and let $u \in H$. Then there is a homomorphism $\chi: H \rightarrow \mathbb{Z}_m$ such that, for all $h \in H \setminus \{e\}$, there exists $\varphi \in \text{End}(H)$ with $\varphi(u) = u$ and $\chi(\varphi(h)) \neq 0$.*

Proof. We prove the claim first for abelian p -groups of the form $\mathbb{Z}_{p^{n_1}} \times \mathbb{Z}_{p^{n_2}}$, then for arbitrary finite abelian p -groups, and finally for arbitrary finite abelian groups.

Assume that $H = \mathbb{Z}_{p^{n_1}} \times \mathbb{Z}_{p^{n_2}}$ with $n_1 \leq n_2$, and write $u = (u_1, u_2)$. Factorize u_i as $a_i p^{k_i}$, where $k_i \leq n_i$ and $p \nmid a_i$. Then u_i has order p^{d_i} in $\mathbb{Z}_{p^{n_i}}$, where $d_i := n_i - k_i$. Let $\mu: \mathbb{Z}_{p^{n_1}} \rightarrow \mathbb{Z}_{p^{n_2}}$ be the homomorphism $\mu(x) = p^{n_2-n_1}x$. We first show that we can assume with no loss of generality that $d_1 \leq d_2$. We use the automorphism $\sigma: H \rightarrow H$ given by $\sigma((x, y)) = (x, \mu(x) + y)$. If $d_1 > d_2$, then

$$\begin{aligned} \mu(u_1) + u_2 &= p^{n_2-n_1}u_1 + u_2 = p^{n_2-n_1}a_1p^{k_1} + a_2p^{k_2} \\ &= a_1p^{n_2-d_1} + a_2p^{n_2-d_2} = p^{n_2-d_1}(a_1 + a_2p^{d_1-d_2}), \end{aligned}$$

with $p \nmid (a_1 + a_2p^{d_1-d_2})$; so the image $v = (u_1, \mu(u_1) + u_2)$ of u under the automorphism σ is such that its second coordinate has order p^{d_1} in $\mathbb{Z}_{p^{n_2}}$, equal to the order of its first coordinate.

With the factorization of H thus adjusted, we now take $\chi: H \rightarrow \mathbb{Z}_{p^{n_2}}$ to be the second projection. Let $h = (h_1, h_2) \in H \setminus \{\hat{0}\}$. If $h_2 \neq 0$, then we can choose φ to be the identity endomorphism of H . Assume now that $h_2 = 0$ and $h_1 \neq 0$. Define $d := d_2 - d_1 \geq 0$, let b be the multiplicative inverse of a_2 in $\mathbb{Z}_{p^{n_2}}$, and let $c := (a_1p^d - a_2)b$. Define $\varphi: H \rightarrow H$ by $\varphi((x, y)) = (x, \mu(x) - cy)$. Obviously $\varphi \in \text{End}(H)$ and $\chi(\varphi(h)) = \mu(h_1) \neq 0$, as μ is injective. It remains to check that $\varphi(u) = u$ or, equivalently, that $\mu(u_1) - cu_2 = u_2$. In fact,

$$\begin{aligned} \mu(u_1) - cu_2 &= p^{n_2-n_1}a_1p^{k_1} - (a_1p^d - a_2)ba_2p^{k_2} \\ &= a_1(p^{n_2-n_1+k_1} - p^{d+k_2}) + a_2p^{k_2} = 0 + u_2, \end{aligned}$$

where the 0 term arises as $n_2 - n_1 + k_1 = n_2 - d_1 = (n_2 - d_2) + (d_2 - d_1) = k_2 + d$.

Next, we prove the claim for abelian p -groups. Assume $H = \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_k}}$, where $n_1 \leq \cdots \leq n_k$. Write $u = (u_1, \dots, u_k)$, let p^{d_i} be the order of u_i in $\mathbb{Z}_{p^{n_i}}$, and define $d := \max(d_1, \dots, d_k)$. If $d \neq d_k$, then pick $i < k$ with $d_i = d$ and consider $\mathbb{Z}_{p^{n_i}} \times \mathbb{Z}_{p^{n_k}}$. By the argument for the previous case, we can find an automorphism of $\mathbb{Z}_{p^{n_i}} \times \mathbb{Z}_{p^{n_k}}$ that sends (u_i, u_k) to (u_i, v) , where v is of order p^d in $\mathbb{Z}_{p^{n_k}}$. Thus by revising the decomposition of H , we can assume with no loss of generality that $d = d_k$. Now take $\chi: H \rightarrow \mathbb{Z}_{p^{n_k}}$ to be the k th projection. Let $h = (h_1, \dots, h_k) \in H \setminus \{\hat{0}\}$. If $h_k \neq 0$, then we can choose φ to be the identity endomorphism of H . So assume that $h_k = 0$. Choose $i < k$ with $h_i \neq 0$. By the argument for the previous case, we can find $\psi \in \text{End}(\mathbb{Z}_{p^{n_i}} \times \mathbb{Z}_{p^{n_k}})$ such that $\psi((u_i, u_k)) = (u_i, u_k)$ and $\pi_2 \circ \psi((h_i, 0)) \neq 0$. If we define φ to act as ψ on $\mathbb{Z}_{p^{n_i}} \times \mathbb{Z}_{p^{n_k}}$ and as the identity on the other factors, then we get our desired endomorphism.

Finally, we prove the claim for arbitrary finite abelian groups. Let p_1, \dots, p_k be distinct primes and let $H = H_1 \times \cdots \times H_k$, where H_i is an abelian p_i -group. Write $u = (u_1, \dots, u_k)$. By the previous case, for each $i \in \{1, \dots, k\}$ we can find a homomorphism $\chi_i: H_i \rightarrow \mathbb{Z}_{p_i^{n_i}}$, where $p_i^{n_i} \mid m$, such that for every $h \in H_i \setminus \{\hat{0}\}$ there exists $\varphi \in \text{End}(H_i)$ satisfying $\varphi(u_i) = u_i$ and $\chi_i(\varphi(h)) \neq 0$. We can now take $\chi := \chi_1 \sqcap \cdots \sqcap \chi_k: H \rightarrow \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$ to be the natural product map. \square

While the following basic lemma can be proved using elementary methods, it also follows immediately from the fact that the cyclic group \mathbb{Z}_m is strongly self-dualizing; see [2, 4.4.2].

Lemma. *Let $m, k \in \mathbb{N}$ and let H be a subgroup of $(\mathbb{Z}_m)^k$. Then H can be described as the set of solutions in \mathbb{Z}_m to a system of homogeneous linear equations in k variables with integer coefficients.*

Proposition. *Assume M is a $j \times k$ matrix over \mathbb{Z}_m whose rows form a subgroup of $(\mathbb{Z}_m)^k$, whose columns form a coset of a subgroup of $(\mathbb{Z}_m)^j$, and which is such that every row contains at least one 0. Then some column is constantly 0.*

Proof. Let H be the subgroup of $(\mathbb{Z}_m)^k$ consisting of the rows of M . Let E be the set of all $\mathbf{c} = (c_1, \dots, c_k) \in \mathbb{Z}^k$ such that the equation “ $\sum_{i=1}^k c_i x_i = 0$ ” is satisfied by every member of H . Define $R = \{ \sum_{i=1}^k c_i \mid \mathbf{c} \in E \}$ and note that $m \in R$ (as H satisfies the equation $mx_1 = 0$). Thus we can define $d = \gcd(R)$.

Case 1: $d > 1$. Choose a prime $p \mid d$. Then $p \mid m$, so $a := m/p$ is a nonzero element of \mathbb{Z}_m . Thus (a, a, \dots, a) is a solution of every homogeneous linear equation satisfied by H , by the choice of p , and hence $(a, a, \dots, a) \in H$, by the previous lemma. But this contradicts the assumption that every row of M contains at least one 0.

Case 2: $d = 1$. Since E is closed under integer linear combinations, there exists $\mathbf{c} \in E$ such that $\sum_{i=1}^k c_i = 1$. Let C_1, \dots, C_k denote the columns of M and define $D = \sum_{i=1}^k c_i C_i$. Then D is a column of M , since we are assuming the columns of M form a coset of a subgroup of $(\mathbb{Z}_m)^j$. But the fact that $\mathbf{c} \in E$ implies that D is constantly 0. \square

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